Quasilinear structures in stochastic arithmetic and their application

Rene Alt [⊗], Jean-Luc Lamotte[⊗], Svetoslav Markov[⊗⊗]
[⊗]Laboratoire Informatique Paris 6 University Pierre et Marie Curie 4 place Jussieu, 75252 Paris cedex 05, France
E-mail: Rene.Alt@lip6.fr
[⊗⊗]Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
"G. Bonchev" st., bl. 8, 1113 Sofia, Bulgaria
E-mail: smarkov@bio.bas.bg

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Aim and Contents

Aim: Study "quasilinear" algebraic structures based on stochastic numbers, operations addition and multiplication by scalar;
Introduce a new arithmetic operation, called "inner stochastic addition" , which is a stochastic analogue to the inner interval addition used in interval arithmetic and interval analysis.

Contents

- Stochastic numbers and Stochastic Arithmetic, definitions
- Arithmetical operations (uncorrelated variables)
- Stochastic arithmetic, algebraic properties
- Case of correlated variables
- Inner operations
- Algebraic properties of inner operations.
- Numerical experiments
- Conclusion and future work.

Stochastic numbers and Stochastic Arithmetic

- Model for exact computation on imprecise data (as is Interval Arithmetic)
- Imprecise data $\Leftrightarrow$ Gaussian distribution = Stochastic number
- The set of stochastic numbers denoted $\mathbb{S}$ is the set of Gaussian random variables.
- An element $X \in \mathbb{S}$ is $X = (m, s)$, m mean value of X, s standard deviation.
- For $X = (m, s)$ then λ_η exists such that

$$P\left(X \in [m - \lambda_\eta s, m + \lambda_\eta s]\right) = 1 - \eta$$

  $I_{\eta,X}$ is the confidence interval of m with a probability $(1 - \eta)$.
  For $\eta = 0.05, \lambda_\eta = 1.96$
- Number of significant digits on m :

$$C_{\eta,X} = log_{10}\left(\frac{|m|}{\lambda_\eta s}\right)$$

Stochastic Arithmetic, operations

- $X = (m, s)$
- Standard stochastic operators

$$X \oplus Y \stackrel{def}{=} \left(m_x + m_y,\ \sqrt{s_x^2 + s_y^2}\right)$$

$$X \ominus Y \stackrel{def}{=} \left(m_x - m_y,\ \sqrt{{s_x}^2 + {s_y}^2}\right)$$

$$X \otimes Y \stackrel{def}{=} \left(m_x m_y,\ \sqrt{{m_y}^2 {s_x}^2 + {m_x}^2 {s_y}^2 + s_x^2\ s_y^2}\right)$$

$$X \oslash Y \stackrel{def}{=} \left(m_x/m_y, \sqrt{\left(\frac{s_x}{m_y}\right)^2 + \left(\frac{m_x s_y}{m_y^2}\right)^2}\right), \text{ with } m_y \neq 0$$

$$\gamma\ X \stackrel{def}{=} (m, |\gamma|\ s)$$

- The first three formulae are exact. The formula for the division is only defined to the first order terms in $\frac{s}{m}$.

Stoch. arithm.: addition and multiplication by Scalars

- Multiplication by -1 (negation) is

$$\neg X = -1 * X = -1 * (m; s) = (-m;\ s).$$

-

$$X_1 \ominus X_2 = X_1 \oplus (\neg X_2) = (m_1; s_1) \oplus \neg (m_2; s_2) = \left(m_1 - m_2;\ \sqrt{s_1^2 + s_2^2} \right).$$

- In particular,
  $X \ominus X = (m; s) \oplus \neg (m; s) = (m - m;\ \sqrt{s^2 + s^2}) = (0; \sqrt{2}s)$.
  The operation $X_1 \ominus X_2$ will be further referred as (outer) subtraction.
- From the definition of addition: $(m; 0) \oplus (0; s) = (m; s)$ showing that every stochastic number can be decomposed into two special stochastic numbers, called resp. degenerate and symmetric.

Stochastic Arithmetic, algebraic properties extended

1. (Markov, Lamotte, Alt, 2004)
   - Monoid w r t addition $\longrightarrow$ extended to group structure with neuter element $(0, 0)$
   - **Group structure $\longrightarrow$ stochastic numbers with negative standard deviations (improper elements) and operations on them.**
   - Multiplication by scalar $\longrightarrow$ s-space.
     • Note that $-1 * X = (-m, s) = \neg X$
   - Computations in s-spaces are reduced to computations in vector spaces.
2. **In this work we remain in the monoid structure and introduce new useful arithmetic operations.**

Case of non independent variables

- Case of correlated variables
  Correlation coefficient ρ

$$X \oplus Y \stackrel{def}{=} \left(m_x + m_y,\ \sqrt{s_x^2 + s_y^2 + 2\rho\, s_x s_y}\right) \qquad (1)$$

- Standard deviation, special cases

$\rho = 1$ $\qquad \sqrt{s_x^2 + s_y^2 + 2s_x s_y} = s_x + s_y$

$\rho = -1$ $\qquad \sqrt{s_x^2 + s_y^2 - 2s_x s_y} = |s_x - s_y|$

$\rho = 0$ $\qquad \sqrt{s_x^2 + s_y^2}$

$\rho = -min(\frac{s_x}{s_y}, \frac{s_y}{s_x})$ $\qquad \sqrt{|s_x^2 - s_y^2|}$

Case of non independent variables (2)

- In many cases the terms are not independent.
- Note that $X \oplus X = (2m, \sqrt{2}s) \neq 2 * X = (2m, 2s)$. because addition supposes that the two variables are independent which is not the case for $X \oplus X$.
- Looking at the sum of two correlated variables, the variance is $s_x^2 + s_y^2 + 2\rho s_x s_y$ which in the case of $Y = X$ leads to $4s^2$ and thus the result is $2X$.
- When computing a single variable polynomial the terms $x, x^2, x^3, \ldots$ are correlated. The sign of ρ depends on the value of x greater or lower than 1
- Example: Value of s_z with $Z = X - X^3/3 + X^5/5 - X^7/7 \quad s_x = 0.01$

ρ	-1	$+1$	0	$-min(\frac{s_x}{s_y}, \frac{s_y}{s_x})$	$exact$
$m_x = 0.5$	**0.0082**	0.0118	0.0101	0.0099	0.0077
$m_x = 1.5$	0.0234	**0.0887**	0.0513	0.0376	0.0733

Inner operations

- The inner addition and inner subtraction operations are defined by

$$X \oplus^- Y = (m_x; s_x) \oplus^- (m_y; s_y) \stackrel{def}{=} (m_x + m_y;\ \sqrt{|s_x^2 - s_y^2|}\ ), \quad (2)$$

$$X \ominus^- Y = (m_x; s_1) \ominus^- (m_y; s_y) \stackrel{def}{=} (m_x - m_y;\ \sqrt{|s_x^2 - s_y^2|}\ ). \quad (3)$$

- Inner operations corresponds to $\rho = -min(\frac{s_x}{s_y}, \frac{s_y}{s_x})$
- They are analogous to inner operations for intervals.
- **Notation.** For $X_1, X_2 \in \mathbb{S}$ denote $X_1 \oplus^+ X_2 = X_1 \oplus X_2$, $X_1 \ominus^+ X_2 = X_1 \ominus X_2$. Then using the binary symbol $\sigma \in \{+, -\}$ we can write $X_1 \oplus^\sigma X_2$, $X_1 \ominus^\sigma X_2$ to denote all four outer and inner operations.
- Outer/inner addition and subtraction are related by $X_1 \oplus^\sigma X_2 = X_1 \ominus^\sigma (\neg X_2)$, $X_1 \ominus^\sigma X_2 = X_1 \oplus^\sigma (\neg X_2)$, $\sigma \in \{+, -\}$.

Abstract definition of inner operations

The *inner* operations for addition/subtraction of (one-dimensional) stochastic numbers can be introduced using an "algebraic approach".

- For $A = (m_A, s_A),\ B = (m_B, s_B) \in \mathbb{S}$, we have

$$A \oplus^- B = X \Longleftrightarrow \begin{cases} Y \ominus B = A, & \text{if solution } Y \text{ exists;} \\ X \ominus A = B, & \text{if solution } X \text{ exists,} \end{cases} \quad (4)$$

- This can be alternatively written as:

$$A \oplus^- B = \begin{cases} Y|_{Y \ominus B = A}, & \text{if } s_B \leq s_A, \\ X|_{X \ominus A = B}, & \text{if } s_A \leq s_B. \end{cases} \quad (5)$$

- Note that the relation $A \oplus^- B = A \ominus^- (\neg B)$ can be deduced from (5)

- Inner subtraction can be introduced similarly.

$$A \ominus^- B = X \Longleftrightarrow \begin{cases} B \oplus Y = A, & \text{if solution } Y \text{ exists;} \\ A \ominus X = B, & \text{if solution } X \text{ exists.} \end{cases} \quad (6)$$

An equivalent way to express the above is:

$$A \ominus^- B = \begin{cases} Y|_{B \oplus Y = A}, & \text{if } s_B \leq s_A; \\ X|_{A \ominus X = B}, & \text{if } s_A \leq s_B. \end{cases} \quad (7)$$

- From (5), (7) we see that inner operations are induced by outer ones. Operations "$\ominus$", "$\ominus^-$", "$\oplus^-$" are deduced from the basic operations "$\oplus$", "$\neg$" (or from "$\oplus$", "$*$"). Therefore we shall further assume that all above mentioned arithmetic operations naturally belong to the algebraic system $(\mathbb{S}, \oplus, \neg)$ (or to system $(\mathbb{S}, \oplus, *)$).
(We may fully write $(\mathbb{R}^+, \oplus) = (\mathbb{R}^+, \oplus, 0, \neg, \oplus^-, \ominus, \ominus^-, \leq)$.

Properties of inner operations

- The stochastic number $X = (0, 0) = 0$ is the unique identity (neutral element) with respect to inner addition $\oplus^-$.
  $\forall A \in \mathbb{S}$:

$$A = 0 \oplus^- A = A \oplus^- 0.$$

- Note that $A \oplus^- (\neg A) = 0$, thus every element $A \in \mathbb{S}$ has unique opposite w. r. t. "$\oplus^-$", and this is the element $\neg A = (-m_A; s_A)$.
- $(0, 0)$ is a particular case of $\underline{0} \ = (m, s)$ with $m \ \leq |s|$

Symmetric stochastic numbers

Consider now the set of symmetric stochastic numbers, i.e. the set of standard deviations with operations $\oplus^-, \ominus^-$ and multiplication by scalar.

- Inner addition "$\oplus^-$" is a closed (total) operation.
- Neutral element, such that $A \oplus^- 0 = A$ for all $A \in \mathbb{R}^+$
- Associativity property $(A \oplus^- B) \oplus^- C = A \oplus^- (B \oplus^- C)$ fails, indeed, e. g. $(7 \oplus^- 5) \oplus^- 3 \neq 7 \oplus^- (5 \oplus^- 3)$, as $(7 \oplus^- 5) \oplus^- 3 = \sqrt{15}$ and $7 \oplus^- (5 \oplus^- 3) = \sqrt{33}$;
- Cancellation $A \oplus^- X = B \oplus^- X \Longrightarrow A = B$ fails. ex: $A = 3, B = 5, X = \sqrt{17}$. Then $3 \oplus^- \sqrt{17} = 5 \oplus^- \sqrt{17}$, but $3 \neq 5$;
- Commutativity: $A \oplus^- B = B \oplus^- A$ holds true.

Conditional associativity of symmetric stoch. numbers

- Associativity holds true under the requirement that the element participating in both brackets is the largest one. We call this property "conditional associativity" or "C-associativity".
  For $B \geq A, B \geq C$ we have
  $(A \oplus^- B) \oplus^- C = A \oplus^- (B \oplus^- C) = \sqrt{|A^2 - B^2 + C^2|}$.
- C-associativity is practically important. In practice this requirement is not too restrictive.
- Cancellation law $A \oplus^- X = B \oplus^- X \Longrightarrow A = B$ fails in $(\mathbb{R}^+, \oplus^-)$. But it holds true unless $X \neq (A \oplus B)/\sqrt{2}$. This is again not too restrictiveSummarizing we obtain:
- So: the set of standard deviations with inner addition $(\mathbb{R}^+, \oplus^-)$, is a c-associative and c-cancellative commutative unital magma.
- Inner addition plays a role in $(\mathbb{R}^+, \oplus)$ analogous to the role of subtraction in the group $(\mathbb{R}, +)$.

Addition and subtraction of symmetric stoch. numbers

- Operations "$+$" and "$\oplus^-$" are closely related when solving equations.
- **Proposition**
  i) For $A, B \in \mathbb{R}^+$, such that $A \leq B$, the unique solution of $A \oplus X = B$ is $X = B \oplus^- A$.
  ii) Equation $A \oplus^- X = B$ has a solution $X = A \oplus B$ for $A, B \in \mathbb{R}^+$.
  If $A, B \in \mathbb{R}^+$ are such that $A \geq B > 0$, then equation $A \oplus^- X = B$ has one solution $X = A \oplus^- B$.
- Solutions of both $A \oplus X = B$ and $A \oplus^- X = B$ become possible under certain conditions. This property is further called "conditional subtractability", briefly "c-subtractability".

C-associativity in the extended monoid system

- Addition and inner addition tightly complement each other in the associative-like properties in the extended monoid system $(\mathbb{R}^+, \oplus, \oplus^-)$.
- Define the mapping $\sigma : \mathbb{R}^{+2} \longrightarrow \{+, -\}$ by

$$\sigma(A, B) = \begin{cases} +, & \text{if } A \geq B; \\ -, & \text{otherwise.} \end{cases}$$

- Then

$$(A \oplus B) \oplus^- C = A \oplus^{\sigma(B,C)} (B \oplus^- C). \tag{8}$$

- **Proposition**. For each triple $A, B, C \in \mathbb{R}^+$ and each pair $\sigma_1, \sigma_2 \in \{+, -\}$, there exist a pair $\sigma_3, \sigma_4 \in \{+, -\}$, such that

$$(A \oplus^{\sigma_1} B) \oplus^{\sigma_2} C = A \oplus^{\sigma_3} (B \oplus^{\sigma_4} C). \tag{9}$$

C-associativity in the extended monoid system

- Example. For $A, B \in \mathbb{R}^+$, $(A \oplus B) \oplus^- A = B$. $(A \oplus B) \oplus^- A = (B \oplus A) \oplus^- A = (B \oplus A) \oplus^- A = (B \oplus A) \oplus^- A = B \oplus^{\sigma(A,A)} (A \oplus^- A) = B \oplus^{\sigma(A,A)} 0 = B$.
- The conditionally associative rules (c-associative rules), give specific conditions under which "replacement of brackets" can be performed; thereby these conditions are easily programmable.
- Outer addition is commutative and associative but has no inverse
- Inner addition is commutative, not associative and has inverse.
- Considering outer and inner addition together, as a "directed" operation in two modes; we can say that this directed operation is c-associative. Thus both modes complement each other.

- Outer addition in $(\mathbb{R}, +) = (\mathbb{R}, +, 0, -, \leq)$ is isotone w. r. t. order relation "$\leq$".
  For standard deviations $X, X_1, C \in \mathbb{R}^+$ we have
  $X \leq X_1 \implies X \oplus C \leq X_1 \oplus C$.
- *Inverse isotonicity of addition.*
  If $A, B, C \in \mathbb{R}^+$, then

$$C \oplus A \leq C \oplus B \Longrightarrow A \leq B,$$

  in particular $C \oplus A = C \oplus B \Longrightarrow A = B$ (cancellation law).
- *Conditional inclusion isotonicity w. r. t. inner addition.*
  For $X, X_1, Y, Y_1 \in \mathbb{R}^+$. If $X \geq X_1, Y \leq Y_1$, then

$$\text{if } X \leq Y, \quad \text{then } X \oplus^- Y \leq X_1 \oplus^- Y_1,$$
$$\text{if } \quad X_1 \geq Y_1, \quad \text{then } X \oplus^- Y \geq X_1 \oplus^- Y_1.$$

A software for computing with stochastic numbers

- Written in Fortran90 and in C++
- New type "Double_st" defined.
- All arithmetic operations and comparisons operators have been overloaded by the corresponding stochastic operators
- Inner addition and subtraction have been added.
- The assigment operator and input output have been redefined
- The standard mathematical functions have been redefined.
- Thus there is two ways for computing with stochastic numbers:

  • The above double precision software emulating the computation with (exact) stochastic numbers.

  • A monte Carlo type computation with discrete stochastic numbers generating N random samples in the range of each data and computing the mean-value and standard deviation of all intermediate result and of the final result.

Experiment, solution of linear systems

- Gaussian elimination with outer and inner addition/subtraction
- Comparison with Monte Carlo simulation (1000 samples)
- Comparison with the Cadna software
- system :

$$\begin{pmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 23 \\ 32 \\ 33 \\ 31 \end{pmatrix}$$

- Exact solution $X = (1, 1, 1, 1)^t$

Experiment, solution of linear systems

- Standard deviation on left an right handsides $s = 0.0001$
  Computed Solution

Outer St. Arithm.		Inner St. Arithm.		Cadna
value	Std. dev.	value	Std. dev.	Value with exactdig.
1.0000	0.7397	1.0000	0.0192	$0.99E + 000$
0.9999	0.5126	0.9999	0.0311	$0.10E + 001$
0.9999	0.0813	0.9999	0.0144	$0.10E + 001$
1.0000	0.0478	1.0000	0.0088	$0.999E + 000$

- Monte Carlo simulation

MonteCarlo	
value	Std. dev.
1.0015	0.0186
1.0006	0.0113
1.0008	0.0048
1.0011	0.0030

- All series of results are in perfect agreement.

Monte Carlo computation of the solution

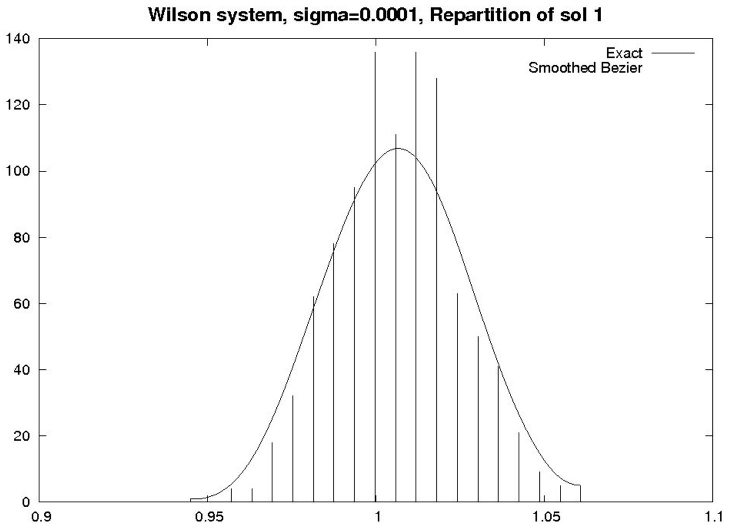

Figure: Distribution of the first component of wilson's system solution

Conclusion

- Stochastic arithmetic has many interesting algebraic structures.
- Leads to results with their confidence intervals
- Interesting for problems with imprecise data.
- Very easy to use in practice.
- Possible computation with outer or inner operations.
- Inner operations take into account some correlation between variables
- Experiments with polynomials and with linear systems show that inner operations give tighter standard deviations for the solution.

To be done

- Develop the theory for multiplication and division.
- Study when to use inner or outer operations.
- Experiment the non linear case
- Make benchmarks on the computing time. The overloading of operations seems much faster than discrete stochastic arithmetic.

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