Pivot Tightening for Direct Methods for Solving Systems of Linear Interval Equations

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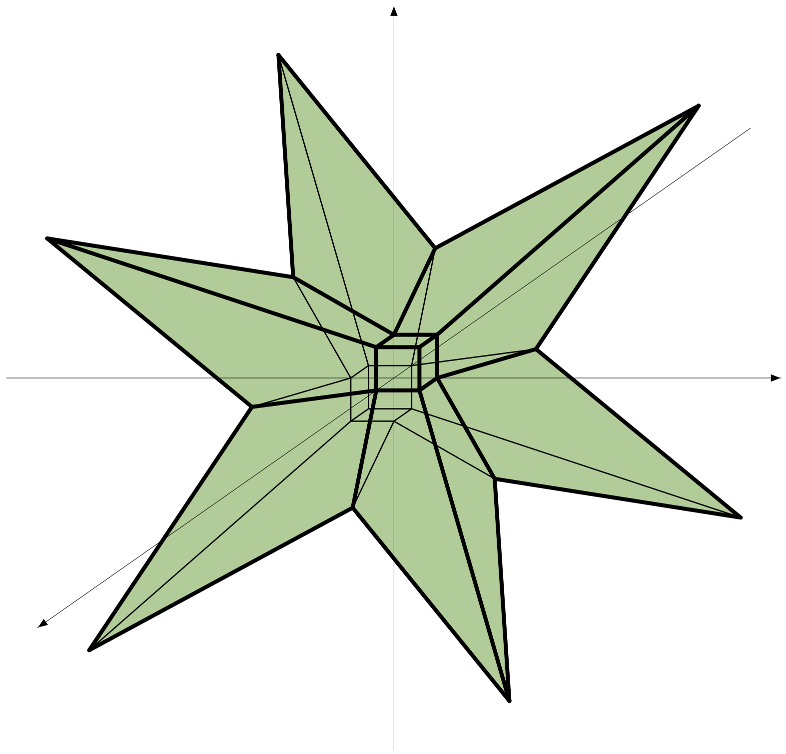
Systems of linear interval equations $[A]x = [b]$

solution set $\quad \Sigma\,([A],[b]) \;:=\; \{x \in \mathbb{R}^n \mid Ax = b,\; A \in [A],\; b \in [b]\}$

General assumption: All $A \in [A]$ are nonsingular.

Then: $\Sigma\,([A],[b])$ is compact, connected, and convex in each orthant.



Neumaier system

$$\begin{pmatrix} 3.5 & [0,2] & [0,2] \\ [0,2] & 3.5 & [0,2] \\ [0,2] & [0,2] & 3.5 \end{pmatrix} x = \begin{pmatrix} [-1,1] \\ [-1,1] \\ [-1,1] \end{pmatrix}$$

(Interval) Hull of the solution set

$$[A]^H[b] \;:=\; \square\Sigma\left([A],[b]\right) \;=\; \bigcap_{\substack{[z]\in\mathbb{IR}^n \\ \Sigma([A],[b])\subseteq[z]}} [z]$$

Interval Gaussian Elimination (abbr. IGE, without pivoting)

To transform a given $[A] \in \mathbb{IR}^{n\times n}$ on triangular form proceed as follows: define $[A]^{(k)} = ([a_{ij}]^{(k)}) \in \mathbb{IR}^{n\times n}$, $k = 1, \ldots, n$, by

$$[A]^{(1)} = [A],$$

$$[a_{ij}]^{(k+1)} = \begin{cases} [a_{ij}]^{(k)}, & i = 1, \ldots, k,\ j = 1, \ldots, n, \\ [a_{ij}]^{(k)} - \frac{[a_{ik}]^{(k)} \cdot [a_{kj}]^{(k)}}{[a_{kk}]^{(k)}}, & i = k+1, \ldots, n,\ j = k+1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that no rows or columns are permuted. The algorithm is feasible if and only if $0 \notin [a_{kk}]^{(k)}$, $k = 1, \ldots, n$.

Further outline

- interval pivot tightening for Gaussian elimination
- application to
  - inverse nonnegative matrices
  - inverse M-matrices
- the interval Cholesky method and pivot tightening
- application to
  - positive definite matrices
  - positive definite Toeplitz matrices

Notation

$\mathbb{IR}$: set of the compact, nonempty real intervals $[a] = [\underline{a}, \overline{a}]$, $\underline{a} \le \overline{a}$,
$\mathbb{IR}^n$: set of n-vectors with components from $\mathbb{IR}$, *interval vectors*
$\mathbb{IR}^{n\times n}$: set of n-by-n matrices with cmpts. from $\mathbb{IR}$. *interval matrices*

Elements from $\mathbb{IR}^n$ and $\mathbb{IR}^{n\times n}$ may be regarded as vector intervals and matrix intervals, respectively, w.r.t. the usual entrywise partial ordering, e.g.,

$$\begin{aligned} [A] &= \Big([a_{ij}]\Big)_{i,j=1}^n = \Big([\underline{a}_{ij}, \overline{a}_{ij}]\Big)_{i,j=1}^n \\ &= [\underline{A}, \overline{A}], \quad \text{where } \underline{A} = \Big(\underline{a}_{ij}\Big)_{i,j=1}^n, \ \overline{A} = \Big(\overline{a}_{ij}\Big)_{i,j=1}^n. \end{aligned}$$

A *vertex matrix* of $[A]$ is a matrix $A = (a_{ij})_{i,j=1}^n$ with $a_{ij} \in \{\underline{a}_{ij}, \overline{a}_{ij}\}$, $i, j = 1, \ldots, n$.
A degenerate interval (vector, matrix) is identified with the (only) real number (vector, matrix) which it contains.

Interval pivot tightening

IGE breaks down when $[a_{kk}]^{(k)}$ (termed *ordinary* interval pivot henceforth) contains zero. This can occur even though all matrices $A \in [A]$ have nonvanishing leading principal minors. For ordinary GE, the pivot $a_{kk}^{(k)}$ can be represented as

$$a_{kk}^{(k)} = \frac{\det A[\{1,2,\ldots,k\}]}{\det A[\{1,2,\ldots,k-1\}]}.$$

This property is used by J. Mayer (2002) to tighten the ordinary interval pivot $[a_{kk}]^{(k)}$: Let D_k denote an enclosure for $\det A[\{1,2,\ldots,k\}]$ for all $A \in [A]$, where $D_0 := 1$. If $0 \notin D_k$, $k = 1,2,\ldots,n$, then $[a_{kk}]^{(k)}$ is replaced by

$$[a_{kk}]^{(k)} \cap \frac{D_k}{D_{k-1}}.$$

Aim:

Finding the range of

$$\frac{\det A[\{1,2,\ldots,k\}]}{\det A[\{1,2,\ldots,k-1\}]}$$

over $[A]$.

$$\text{w.l.o.g.} \qquad p(A) := \frac{\det A}{\det A'},$$

$$\text{where} \quad A' := A\left[\{1,2,\ldots,n-1\}\right].$$

Inverse nonnegative matrices

Def.: A is *inverse nonnegative* if $0 \le A^{-1}$.

Proposition 1 (Kuttler, 1971). *Let* $[A] = [\underline{A}, \overline{A}]$ *be a matrix interval and* $\underline{A}$ *and* $\overline{A}$ *be inverse nonnegative. Then* $[A]$ *is inverse nonnegative and* $\overline{A}^{-1} \le \underline{A}^{-1}$.

Theorem 2 (Beeck, 1975). *If* $[A] \in \mathbb{IR}^{n\times n}$ *is inverse nonnegative, then*

$$[A]^H[b] = \begin{cases} [\overline{A}^{-1}\underline{b},\ \underline{A}^{-1}\overline{b}] & \text{if} \quad 0 \le [b], \\ [\underline{A}^{-1}\underline{b},\ \underline{A}^{-1}\overline{b}] & \text{if} \quad 0 \in [b], \\ [\underline{A}^{-1}\underline{b},\ \overline{A}^{-1}\overline{b}] & \text{if} \quad [b] \le 0. \end{cases}$$

In the general case, one has to solve at most $2n$ linear systems to find $\inf([A]^H[b])$ and similarly $\sup([A]^H[b])$.

We assume now that each $A \in [A]$ has the property that all its leading principal submatrices are inverse nonnegative or equivalently that A allows a factorization $A = LDU$, where L and U are inverse nonnegative lower and upper triangular matrices, respectively, whose diagonal entries are all one, and D is a diagonal matrix with positive diagonal entries. According to Prop. 1 the condition on $[A] = [\underline{A}, \overline{A}]$ is fulfilled if all leading principal submatrices of $\underline{A}$ and $\overline{A}$ are inverse nonnegative.

Theorem 3. *If all leading principal submatrices of $[A] = [\underline{A}, \overline{A}]$ are inverse nonnegative then*

$$p([A]) = [p(\underline{A}), p(\overline{A})]. \tag{1}$$

Proof. For $\underline{A} \le A \le \overline{A}$ it follows from Prop. 1 that

$$(\overline{A}^{-1})_{nn} \le (A^{-1})_{nn} \le (\underline{A}^{-1})_{nn}.$$

Formula (1) is now a consequence of $(A^{-1})_{nn} = \frac{1}{p(A)}$. □

Example

$$\text{Let}\quad [A] := \begin{pmatrix} [4,5] & [-3,-2] & 1 \\ [-3,-2] & 4 & [-3,-2] \\ 1 & [-3,-2] & [4,5] \end{pmatrix}.$$

Since $\underline{A}$, $\overline{A}$ are inverse nonnegative, $[A]$ contains only inverse nonnegative matrices. IGE results in the following interval containing zero

$$[a_{33}]^{(3)} = [-.112\ldots, 4.311\ldots]$$

and breaks down.

Application of Theorem 3 results in $[\frac{6}{7}, 4]$.

Inverse M-matrices

Def.: $A \in \mathbb{R}^{n \times n}$ is called an *inverse M-matrix* if A^{-1} is an M-matrix, i.e., if $0 \leq A$ and the off-diagonal entries of A^{-1} are nonpositive.

Each leading principal submatrix of an inverse M-matrix is likewise an inverse M-matrix and all its principal minors are positive.

Proposition 4 (Johnson and Smith, 2002)**.** *A matrix interval is an inverse M-matrix interval if and only if all its vertex matrices are inverse M-matrices.*

Theorem 5. *If* $[A] \in \mathbb{IR}^{n\times n}$ *is an inverse* M*-matrix interval then* $p([A]) = \left[p(A^l), p(A^u)\right]$, *where*

$$A^l := \begin{pmatrix} \underline{a}_{11} & \cdots & \underline{a}_{1,n-1} & \overline{a}_{1n} \\ \vdots & & \vdots & \vdots \\ \underline{a}_{n-1,1} & \cdots & \underline{a}_{n-1,n-1} & \overline{a}_{n-1,n} \\ \overline{a}_{n1} & \cdots & \overline{a}_{n,n-1} & \underline{a}_{nn} \end{pmatrix}, \quad A^u := \begin{pmatrix} \overline{a}_{11} & \cdots & \overline{a}_{1,n-1} & \underline{a}_{1n} \\ \vdots & & \vdots & \vdots \\ \overline{a}_{n-1,1} & \cdots & \overline{a}_{n-1,n-1} & \underline{a}_{n-1,n} \\ \underline{a}_{n1} & \cdots & \underline{a}_{n,n-1} & \overline{a}_{nn} \end{pmatrix}.$$

Example

$$\text{Let} \quad [A] := \begin{pmatrix} [1,4] & [\frac{1}{2}, \frac{\sqrt{2}}{2}] & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 1 & [\frac{1}{2}, \frac{\sqrt{2}}{2}] \\ \frac{1}{2} & \frac{\sqrt{2}}{2} & 1 \end{pmatrix}.$$

It is easily checked that all eight vertex matrices are inverse M-matrices and by Proposition 4 it follows that $[A]$ is an inverse M-matrix interval. IGE results in the following interval containing zero

$$[a_{33}]^{(3)} = [-\frac{4\sqrt{2}+1}{64}, 1 - \frac{\sqrt{2}}{16}]$$

and breaks down. By Theorem 5 this pivot can be tightened to $[\frac{1}{2}, 1 - \frac{\sqrt{2}}{4}] = [0.5, 0.6464\ldots]$.

Similar results hold for *totally nonnegative matrices*, i.e., matrices having all their minors nonnegative, cf.

G., *Interval Gaussian elimination with pivot tightening*, SIAM J. Matrix Anal. Appl. 30(4), pp. 1761–1772 (2009).

Therein also results on Toeplitz systems of equations and Neville elimination can be found.

Symmetric solution set

Let $[A]$ be symmetric, i.e.,

$$[A]^T = [A],$$

and

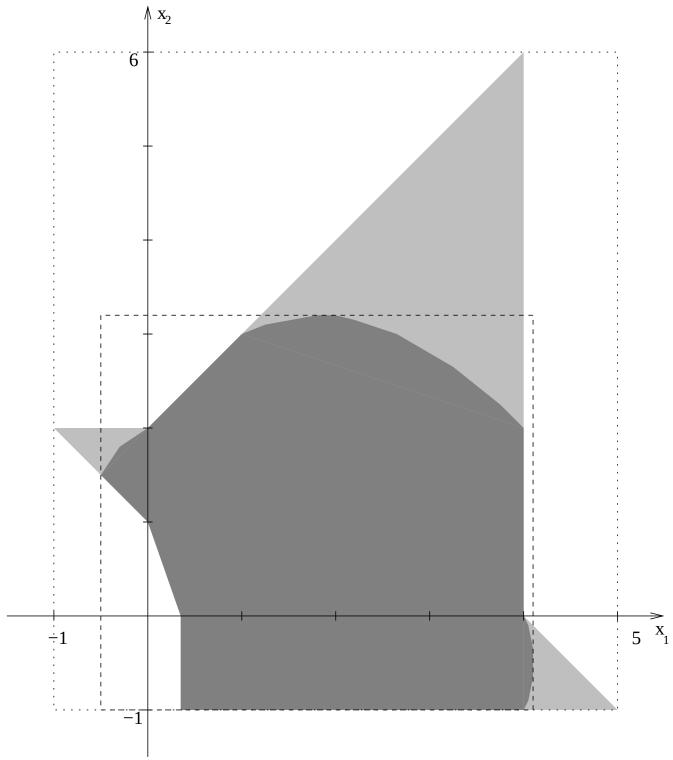
$$[A]_s \;:=\; \left\{ A \in [A] \;\mid\; A = A^T \right\} \quad \text{(\textit{symmetric part} of } [A]\text{)}.$$

Symmetric solution set:

$$\Sigma_{sym}\,([A],[b]) \;:=\; \{x \in \mathbb{R}^n \;\mid\; Ax = b,\, A \in [A]_s,\, b \in [b]\}$$

Symmetric solution set – example

$$\begin{pmatrix} [1,3] & [0,1] \\ [0,1] & [-4,1] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} [1,4] \\ [-2,1] \end{pmatrix}$$



Interval Cholesky method (Alefeld and G. Mayer, 1993, 2009)

Given $[A] = [A]^T \in \mathbb{IR}^{n\times n}$, define the lower triangular matrix $[L]$ with $[A] \subseteq [L][L]^T$ by

$$\begin{aligned}
[l_{jj}] &= \left([a_{jj}] - \sum_{k=1}^{j-1} [l_{jk}]^2\right)^{\frac{1}{2}}, \\
& \qquad\qquad\qquad\qquad\qquad\qquad j = 1, \ldots, n; \\
[l_{ij}] &= \left([a_{ij}] - \sum_{k=1}^{j-1} [l_{ik}][l_{jk}]\right) / \, [l_{jj}], \;\; i = j+1, \ldots, n.
\end{aligned}$$

Here $[a]^{\frac{1}{2}} := \left\{ a^{\frac{1}{2}} \mid a \in [a] \right\}$ for $0 \le \underline{a}$.
The algorithm is feasible if and only if $0 < \underline{l}_{ii}$, $i = 1, \ldots, n$.

Interval pivot tightening

The interval Cholesky method breaks down when $0 \not< \underline{l}_{ii}$. This can occur even though all matrices $A \in [A]_s$ are positive definite. For the ordinary Cholesky decomposition, l_{ii} can be represented as

$$l_{ii} = \left(\frac{\det A[\{1, 2, \ldots, i\}]}{\det A[\{1, 2, \ldots, i-1\}]} \right)^{\frac{1}{2}}.$$

An easy upper bound: By Fischer's inequality

$$\overline{l}_{jj} \leq \overline{a}_{jj}^{\frac{1}{2}}, \quad j = 1, \ldots, n.$$

Positive definite interval matrices

Let $A_c := \frac{1}{2}(\underline{A} + \overline{A})$ *midpoint matrix*
and $\triangle A := \frac{1}{2}(\overline{A} - \underline{A})$ *radius matrix*;
then $[A] = [A_c - \triangle A, A_c + \triangle A]$.
Consider the following vertex matrices (of cardinality of at most 2^{2n-1})

$$A_{yz} := A_c - \text{diag}(y_1, \ldots, y_n) \cdot \triangle A \cdot \text{diag}(z_1, \ldots, z_n), \quad y, z \in \{-1, 1\}^n;$$

in particular, the matrices A_{zz}, $z \in \{-1, 1\}^n$, form a subset of the vertex matrices of cardinality of at most 2^{n-1}.

Proposition 6 (Bialas and G. 1984, Rohn 1994). *Let* $[A] = [A]^T \in \mathbb{IR}^{n\times n}$*. Then* $[A]$ *is positive definite, i.e.,* $[A]_s$ *contains only positive definite matrices if and only if all matrices* A_{zz}*,* $z \in \{-1, 1\}^n$*, are positive definite.*

Conjecture:

For a symmetric positive definite interval matrix $[A]$ the minimum value of $p = \det A / \det A'$ over $[A]_s$ is attained at a matrix A_{zz}.

Let $A \in \mathbb{R}^{n\times n}$ be symmetric and order its eigenvalues as $\lambda_1(A) \leq \ldots \leq \lambda_n(A)$. By the interlacing of the eigenvalues of A and $A' = A[\{1,\ldots,n-1\}]$, we obtain

$$\lambda_1(A) \ \leq \ p(A) \ = \ \frac{\det A}{\det A'}.$$

For $[A] = [A]^T \in \mathbb{IR}^{n\times n}$

$$\min \{\lambda_1(A) \mid A \in [A]_s\} \ = \ \min \{\lambda_1(A_{zz}) \mid z \in \{-1,1\}^n\}$$

(Rohn, 1998).

Lower bounds for the smallest eigenvalue of a symmetric pd matrix

Let $A = \begin{pmatrix} A' & b \\ b^T & c \end{pmatrix}$ and denote by β_{n-1} a lower bound for $\lambda_1(A')$. Then

Dembo (1988):

$$\beta_n = \frac{1}{2}\left(c + \beta_{n-1} - \sqrt{(c - \beta_{n-1})^2 + 4b^T b}\right) \leq \lambda_1(A)$$

(may not be positive);

Ma and Zarowski (1998):

$$0 < \tilde{\beta}_n = \frac{1}{2}\left(c + \beta_{n-1} - \sqrt{(c - \beta_{n-1})^2 + 4\beta_{n-1} b^T (A')^{-1} b}\right) \leq \lambda_1(A).$$

Example (cont'd)

$[A] = \begin{pmatrix} [4,5] & [-3,-2] & 1 \\ [-3,-2] & 4 & [-3,-2] \\ 1 & [-3,-2] & [4,5] \end{pmatrix}$ is pd but the interval Cholesky method breaks down due to

$$\underline{l}_{33}^2 = -0.112\ldots.$$

The matrices A_{zz}:

	$\begin{pmatrix} 4 & -3 & 1 \\ -3 & 4 & -3 \\ 1 & -3 & 4 \end{pmatrix}$	$\begin{pmatrix} 4 & -3 & 1 \\ -3 & 4 & -2 \\ 1 & -2 & 4 \end{pmatrix}$	$\begin{pmatrix} 4 & -2 & 1 \\ -2 & 4 & -3 \\ 1 & -3 & 4 \end{pmatrix}$	$\begin{pmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{pmatrix}$
Dembo bound:	-1	$-0.192\ldots$	$-0.316\ldots$	$0.550\ldots$
Ma and Zarowski bound:	$0.177\ldots$	$0.658\ldots$	$0.619\ldots$	

Thus $\underline{l}_{33}^2$ can be improved by $0.177\ldots.$

Positive definite Toeplitz matrices

$$T = T(a_1, \ldots, a_n) = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2 \\ a_n & \cdots & a_2 & a_1 \end{pmatrix}$$

Proposition 7 (Mukherjee and Maiti, 1988)**.**

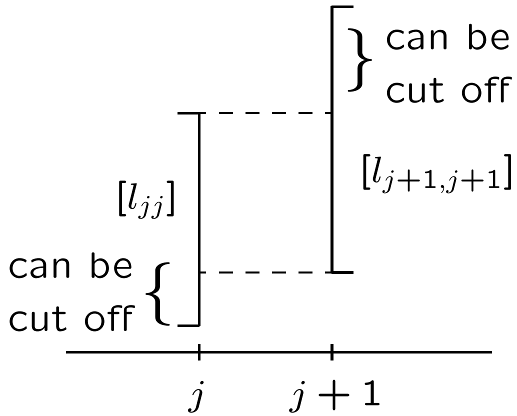
$$l_{nn} \le \ldots \le l_{22} \le l_{11} \le a_1^{\frac{1}{2}}.$$

Practical application to $[l_{jj}] = [\underline{l}_{jj}, \bar{l}_{jj}]$:

If $\quad \bar{l}_{j+1,j+1} > \bar{l}_{j,j} \quad$ then put $\quad \bar{l}_{j+1,j+1} := \bar{l}_{j,j}$;

if $\quad \underline{l}_{j+1,j+1} > \underline{l}_{j,j} \quad$ then put $\quad \underline{l}_{j,j} := \underline{l}_{j+1,j+1}$.

In the second case the entries below $[l_{j,j}]$ should be recomputed; this may tighten $[l_{j+1,j+1}]$, too.



Example: $[T] = T(16, [9, 10], [4, 5], [1, 2], [0, 1])$

diagonal entry	only monotonicity	use of matrices A_{zz}
$[l_{22}]$	$[3.1224, 3.3072]$	
$[l_{33}]$	$[2.8770, 3.4416]$ $\downarrow$ 3.3072	$[3.0382, 3.4416]$ $\downarrow$ 3.3072
$[l_{44}]$	$[2.2309, 3.6190]$ $\downarrow$ 3.3072	$[3.0110, 3.6417]$ $\downarrow$ 3.3072
$[l_{55}]$	$[-7.5673, \quad]$	$[2.9638, 3.8438]$ $\downarrow$ 3.3072

Example (cont'd) $[T] = T(16, [9, 10], [4, 5], [1, 2], [0, 1])$

The entries are nonnegative and exhibit a monotone and convex decay.

Results by Berenhaut and Bandyopadhyay, 2005:

For each $T \in [T]$, positivity and monotonicity are maintained through the Cholesky decomposition $T = LL^T$, where the lower triangular matrix $L = (l_{ij})$ satisfies

$$\begin{aligned}
l_{ij} &\geq 0, \quad 1 \leq j \leq i \leq n; \\
l_{ij} &\geq l_{i+1,j}, \quad 1 \leq j \leq i \leq n-1; \\
l_{jj} &\geq \sqrt{(a_1 - a_2) + a_j(a_{j-1} - a_j)} \quad (\geq 0), \quad 1 \leq j \leq n; \\
l_{ij} &\geq \Big(a_{i-j+1} - a_{i-j+2} + a_i(a_{j-1} - a_j)\Big)/l_{jj}, \quad 1 \leq j \leq i \leq n.
\end{aligned}$$