Enclosures to the solution sets of parametric interval linear systems

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Parametric interval linear systems

Notation

An interval vector

$$\mathbf{p} := [\underline{p}, \overline{p}] = \{p \in \mathbb{R}^n \mid \underline{p} \le p \le \overline{p}\},$$

and the corresponding center and radius vectors

$$p^c := \frac{1}{2}(\overline{p} + \underline{p}), \quad p^\Delta := \frac{1}{2}(\overline{p} - \underline{p}).$$

Parametric interval linear systems

Parametric system of interval linear equations

$$A(p)x = b(p),$$

where

$$A(p) = \sum_{k=1}^{K} p_k A^k, \quad b(p) = \sum_{k=1}^{K} p_k b^k, \quad p \in \mathbf{p},$$

and $A^k \in \mathbb{R}^{n\times n}$, and $b^k \in \mathbb{R}^n$, $k = 1, \dots, K$. The solution set is defined as

$$\Sigma := \{x \in \mathbb{R}^n \mid A(p)x = b(p),\ p \in \mathbf{p}\}.$$

It covers (skew-)symmetric, Toeplitz, Hankel,... systems

Problem

Find a tight enclosure to Σ.

Selected publications

Theoretical papers

- Popova & Krämer, 2008
- Popova, 2002, 2009

Methods

- Rump, 1994
- Kolev, 2002, 2004, 2006
- Popova, 2000, 2007
- Popova & Krämer, 2007
- Skalna, 2006, 2008

Software

- Popova, 2007 (*Mathematica* package)
- Popova & Krämer, 2008

Regularity of parametric interval matrices

Definition (Regularity)

The parametric matrix $A(p)$, $p \in \mathbf{p}$, is *regular* if $A(p)$ is non-singular for each $p \in \mathbf{p}$.

Theorem (sufficient condition)

If

$$\rho\left(\sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k|\right) < 1 \qquad (*)$$

then $A(p)$, $p \in \mathbf{p}$, is regular.

Theorem

Let $A(p^c)$ be nonsingular and $z \in \{\pm 1\}^n$, $y \in \{\pm 1\}^K$ such that $y_k \operatorname{diag}(z) A(p^c)^{-1} A^k \operatorname{diag}(z) \geq 0$. Then $A(\mathbf{p})$ is regular iff and only if $(*)$ holds.

Relaxing and preconditioning (or vice versa?)

- Relaxing and preconditioning by R:

$$\mathbf{A} := R\left(\sum_{k=1}^{K} \mathbf{p}_k A^k\right), \quad \mathbf{b} := R\left(\sum_{k=1}^{K} p_k b^k\right)$$

- Preconditioning by R and relaxing:

$$\mathbf{A}' := \sum_{k=1}^{K} \mathbf{p}_k (RA^k), \quad \mathbf{b}' := \sum_{k=1}^{K} p_k (Rb^k)$$

Observation

We have $\mathbf{A}' \subseteq \mathbf{A}$ and $\mathbf{b}' \subseteq \mathbf{b}$.

Characterization of Σ

Theorem (Necessary condition)

If $x \in \Sigma$ then it solves

$$|A(p^c)x - b(p^c)| \leq \sum_{k=1}^{K} p_k^{\Delta} |A^k x - b^k|.$$

Note

It is necessary and sufficient for a special subclass [Popova, 2009].

Theorem (Necessary and sufficient condition)

We have that $x \in \Sigma$ if and only if it solves

$$y^T (A(p^c)x - b(p^c)) \leq \sum_{k=1}^{K} p_k^{\Delta} |y^T (A^k x - b^k)|$$

for every $y \in \mathbb{R}^n$.

Hansen–Bliek–Rohn (HBR) method

Theorem

Suppose that $A(p^c)$ be nonsingular. Denote

$$M := \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k|, \qquad x^* := A(p^c)^{-1} b(p^c),$$

$$M^* := (I - M)^{-1}, \qquad x^0 := M^* |x^*| + \sum_{k=1}^{K} p_k^{\Delta} M^* |A(p^c)^{-1} b^k|.$$

If $\rho(M) < 1$ then any $x \in \Sigma$ satisfies

$$x_i \le \max\left\{ x_i^0 + (x_i^* - |x^*|_i) m_{ii}^*, \ \frac{1}{2m_{ii}^* - 1}\left(x_i^0 + (x_i^* - |x^*|_i) m_{ii}^*\right) \right\},$$

$$x_i \ge \min\left\{ -x_i^0 + (x_i^* + |x^*|_i) m_{ii}^*, \ \frac{1}{2m_{ii}^* - 1}\left(- x_i^0 + (x_i^* + |x^*|_i) m_{ii}^*\right) \right\}.$$

Bauer–Skeel bounds

Theorem

Suppose that $A(p^c)$ *be nonsingular. Denote*

$$M := \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k|,$$
$$x^* := A(p^c)^{-1} b(p^c).$$

If $\rho(M) < 1$ *then we have an enclosure*

$$\Big[x^* - (I - M)^{-1} \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x^* - b^k)|,$$
$$x^* + (I - M)^{-1} \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x^* - b^k)| \Big].$$

Comparison

Who is the winner?

- For standard interval linear systems, the HBR enclosure is never worse than the Bauer–Skeel one [Rohn, 2010].
- For parametric interval linear systems, the Bauer–Skeel enclosure may be better!

Recommendation

Compute both and intersect them.

Refinement of the HBR bounds

In the proof of HBR enclosure we use estimation

$$|x - x^*| \le \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x - b^k)| \le Z|x| + z.$$

Let $\mathbf{x}$ be an enclosure of Σ and put $\mathbf{a}^k := A(p^c)^{-1}(A^k\mathbf{x} - b^k)$. If $\underline{a}^k \ge 0$ then

$$|A(p^c)^{-1}(A^k x - b^k)| = A(p^c)^{-1}A^k x - A(p^c)^{-1}b^k$$

(similarly if $\overline{a}^k \le 0$), otherwise we use the standard estimation. Thus

$$\begin{aligned}|x - x^*| &\le \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x - b^k)| \\ &\le Yx - y + Z'|x| + z' \ \le (|Y| + Z')|x| - y + z',\end{aligned}$$

Observation

We obtain at least as tight enclosure as the HBR one.

Refinement of the Bauer–Skeel bounds

In the proof of Bauer–Skeel bounds we use estimation

$$|x - x^*| \le \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x - b^k)| \le Z|x - x^*| + z.$$

Let $\mathbf{x}$ be an enclosure of Σ and $\mathbf{a}^k := A(p^c)^{-1}(A^k\mathbf{x} - b^k)$. If $\underline{a}^k \ge 0$ then

$$|A(p^c)^{-1}(A^k x - b^k)| = A(p^c)^{-1}A^k(x - x^*) + A(p^c)^{-1}(A^k x^* - b^k)$$

(similarly if $\overline{a}^k \le 0$), otherwise we use the standard estimation. Thus

$$\begin{aligned} |x - x^*| &\le Y(x - x^*) + y + Z'|x - x^*| + z' \\ &\le (|Y| + Z')|x - x^*| + y + z', \end{aligned}$$

Observation

We obtain at least as tight enclosure as the Bauer–Skeel one.

Examples

Example (Okumura's problem [Popova & Krämer, 2008])

$$\begin{pmatrix} p_1+p_6 & -p_6 & 0 & 0 & 0 \\ -p_6 & p_2+p_6+p_7 & -p_7 & 0 & 0 \\ 0 & -p_7 & p_3+p_7+p_8 & -p_8 & 0 \\ 0 & 0 & -p_8 & p_4+p_8+p_9 & -p_9 \\ 0 & 0 & 0 & -p_9 & p_5+p_9 \end{pmatrix} x = \begin{pmatrix} 10 \\ 0 \\ 10 \\ 0 \\ 0 \end{pmatrix},$$

where $p_i \in [0.99, 1.01]$, $i = 1, \dots, 9$.

Examples

Example (cont.)

- The HBR method gives

$$([6.9693, 7.2150], [4.0689, 4.2971], [5.3501, 5.5612], [2.1083, 2.2568], [1.0397, 1.1431]),$$

- and its refinement yields

$$([6.9925, 7.1913], [4.1134, 4.2504], [5.3799, 5.5307], [2.1324, 2.2317], [1.0576, 1.1244]).$$

- The Bauer–Skeel method gives

$$([7.0148, 7.1671], [4.1173, 4.2463], [5.3933, 5.5158], [2.1377, 2.2260], [1.0601, 1.1217]),$$

- and its refinement yields

$$([7.0151, 7.1667], [4.1180, 4.2456], [5.3938, 5.5153], [2.1382, 2.2255], [1.0605, 1.1213]).$$

- Here, the exact interval hull of Σ is known [Popova & Krämer, 2008]

$$([7.0170, 7.1663], [4.1193, 4.2454], [5.3952, 5.5150], [2.1392, 2.2253], [1.0614, 1.1211]).$$

Examples

Example (Random symmetric systems)

Data were generated randomly as follows:

- A^c_{ij} were chosen randomly and independently in $[-10, 10]$ with uniform distribution;
- symmetrize $A^c := A^c + (A^c)^T + 10nI$;
- $A^\Delta_{ij} := R$, where $R > 0$ is a parameter;
- **b** was chosen to be crisp with entries randomly from $[-10, 10]$.

Note

- Computations in MATLAB with INTLAB.
- Sequence of 10 runs with various settings of a dimension n and radius R.
- Efficiency measured by sum of radii.

Example (cont.)

n	R	sum of radii				average computing time (in sec.)			
		BS	ref. BS	HBR	ref. HBR	BS	ref. BS	HBR	ref. HBR
5	0.05	0.04411	0.04407	0.04803	0.04416	0.01537	0.0893	0.01421	0.0886
5	0.1	0.08454	0.08441	0.09157	0.08446	0.0155	0.09176	0.0148	0.08898
5	0.5	0.4293	0.4247	0.4755	0.4255	0.01553	0.09054	0.01441	0.0911
5	1	1.02	1.003	1.098	1.015	0.01412	0.08438	0.01366	0.08453
10	0.05	0.03669	0.03666	0.04041	0.03674	0.04731	0.5632	0.0456	0.5579
10	0.1	0.06011	0.06	0.06576	0.06013	0.04588	0.5559	0.04498	0.5494
10	0.5	0.3504	0.347	0.3835	0.3496	0.04839	0.5813	0.04604	0.5679
10	1	0.9912	0.9752	1.075	0.9948	0.04638	0.5461	0.04401	0.5449
15	0.05	0.03551	0.03548	0.0389	0.03553	0.1017	1.802	0.1	1.778
15	0.1	0.07195	0.07181	0.07872	0.07197	0.09783	1.719	0.09587	1.695
15	0.5	0.3912	0.3878	0.4269	0.3918	0.09836	1.759	0.09593	1.724
15	1	0.8311	0.8172	0.907	0.8338	0.09666	1.733	0.0956	1.731
20	0.05	0.03385	0.03381	0.03717	0.03385	0.1758	3.979	0.1695	3.956
20	0.1	0.07319	0.07305	0.08081	0.07326	0.1721	3.937	0.1697	3.928
20	0.5	0.4017	0.3979	0.4427	0.4033	0.1726	3.961	0.1671	3.976
20	1	0.793	0.7799	0.8648	0.7986	0.1699	4.01	0.169	3.996
25	0.05	0.03413	0.0341	0.03743	0.03414	0.2774	7.591	0.2647	7.524
25	0.1	0.067	0.06687	0.07359	0.06704	0.283	7.712	0.2775	7.644
25	0.5	0.3948	0.3913	0.4331	0.3966	0.2726	7.599	0.2669	7.493
25	1	0.8265	0.8126	0.9052	0.8345	0.2767	7.723	0.2704	7.77

Fin