Computing Isolated Singular Solutions of Polynomial Systems Accurately: Case of Breadth One

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A Motivating Example [Ojika 1987]

$$F = \{ f_1 = x_1^2 + x_2 - 3,\; f_2 = x_1 + \frac{1}{8}x_2^2 - \frac{3}{2} \}$$

- $\hat{\mathbf{x}} = (1,2)$ is a 3-fold solution since $F(\hat{\mathbf{x}}) = 0$ and

$$\left(\frac{\partial}{\partial x_1} - \frac{2\partial}{\partial x_2} \right) F(\hat{\mathbf{x}}) = 0,$$

$$\left(\frac{\partial}{\partial x_2} - \frac{\partial^2}{2\partial x_1^2} + \frac{2\partial^2}{\partial x_1 \partial x_2} - \frac{2\partial^2}{\partial x_2^2} \right) F(\hat{\mathbf{x}}) = 0.$$

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- Newton's method with $\text{Digits} = 14$ in Maple and an initial solution $[0., 3.]$, after
  - 10 iterations, we get $[0.9920798, 2.0158017]$;
  - 100 iterations, we get $[1.0000363, 1.9999274]$;
  - 1000 iterations, we still get $[1.0000363, 1.9999274]$.

Related Previous Work

- Compute the multiplicity structure of a singular solution
  Zeng'09, Damiano-Sabadini-Struppa'07, Dayton-Zeng'05, Möller-Tenberg'01, Mourrain'96, Möller-Stetter'95, Marinari-Mora-Möller'95.

- Refine an approximate singular solution
  Leykin-Verschelde-Zhao'05,06, Lercerf'03, Ojika-Watanabe-Mitsui'83, Decker-Keller-Kelley'80,83, Griewank-Osborne'81, Reddien'78, Rall'66.

- A special case: Jacobian matrix has corank one
  Giusti-Lercerf-Salvy-Yakoubsohn'07, Dayton-Zeng'05, Griewank'85.

Notations

- Consider a polynomial system $F \in \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$,

$$F : \begin{cases} f_1(x_1, \ldots, x_n) & = & 0, \\ f_2(x_1, \ldots, x_n) & = & 0, \\ & \vdots & \\ f_n(x_1, \ldots, x_n) & = & 0. \end{cases}$$

- Let $I = (f_1, \ldots, f_n)$ be the ideal generated by $f_1, \ldots, f_n$.

- Let $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_n)$ be an isolated singular solution of F with the multiplicity μ.

Max Noether Conditions

- Let $D(\alpha) = D(\alpha_1, \ldots, \alpha_n) : \mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{x}]$ denote the differential operator defined by:

$$D(\alpha_1, \ldots, \alpha_n) := \frac{1}{\alpha_1! \cdots \alpha_n!} \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

  for nonnegative integer array $\alpha = [\alpha_1, \ldots, \alpha_n]$.

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  for nonnegative integer array $\alpha = [\alpha_1, \ldots, \alpha_n]$.

- Let $\mathfrak{D} = \{D(\alpha) \| |\alpha| \geq 0\}$, we define the Max Noether space associated to I at $\hat{\mathbf{x}}$ as

$$\triangle_{\hat{\mathbf{x}}}(I) := \{L \in Span_{\mathbb{C}}(\mathfrak{D}) | L(f)_{\mathbf{x}=\hat{\mathbf{x}}} = 0, \ \forall f \in I\}.$$

  $\forall\ L \ \in \triangle_{\hat{\mathbf{x}}}(I)$ is called a Max Noether condition and $\dim_{\mathbb{C}}(\triangle_{\hat{\mathbf{x}}}(I)) = \mu$.

Closedness Condition

Two morphisms on $\mathfrak{D} = \{D(\alpha) \mid |\alpha| \geq 0\}$

- Anti-differentiation operator

$$\Phi_j(D(\alpha)) := \begin{cases} D(\alpha_1, \ldots, \alpha_j - 1, \ldots, \alpha_n), & \text{if } \alpha_j > 0, \\ 0, & \text{otherwise.} \end{cases}$$

- Differential operator

$$\Psi_j(D(\alpha)) := D(\alpha_1, \ldots, \alpha_j + 1, \ldots, \alpha_n).$$

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Theorem. *[Stetter'04] A subspace $\triangle_{\hat{\mathbf{x}}}$ of $Span_{\mathbb{C}}(\mathfrak{D})$ is said to be closed if and only if its dimension is finite and*

$$L \in \triangle_{\hat{\mathbf{x}}} \Longrightarrow \Phi_j(L) \in \triangle_{\hat{\mathbf{x}}}, \ j = 1, \ldots, n.$$

Case of Breadth One

Lemma. *Suppose corank of the Jacobian matrix* $F'(\hat{\mathbf{x}})$ *is one, i.e.,* $\dim \mathrm{Nullspace}(F'(\hat{\mathbf{x}})) = 1$*and for a nonnegative integer* k*, let* $\triangle_{\hat{\mathbf{x}}}^{(k)}(I) = \{L \in \triangle_{\hat{\mathbf{x}}}(I), order(L) \leq k\}$*, then for* $1 \leq k \leq \mu - 1$*,*

$$\dim(\triangle_{\hat{\mathbf{x}}}^{(k)}(I)) = \dim(\triangle_{\hat{\mathbf{x}}}^{(k-1)}(I)) + 1,$$

and for $k \geq \mu$*,*

$$\dim(\triangle_{\hat{\mathbf{x}}}^{(k)}(I)) = \dim(\triangle_{\hat{\mathbf{x}}}^{(\mu-1)}(I)).$$

Max Noether Conditions in Breadth One Case

Theorem. *[Li and Zhi'09] Suppose* $\mathrm{corank}(F'(\hat{\mathbf{x}})) = 1$, *and* $L_0 = 1_{id}, L_1 = \frac{\partial}{\partial x_1} \in \triangle_{\hat{\mathbf{x}}}^{(1)}(I)$. *We construct* L_k *incrementally for* k *from* 2 *to* $\mu - 1$ *by*

$$L_k = P_k + a_{k,2}\frac{\partial}{\partial x_2} + \cdots + a_{k,n}\frac{\partial}{\partial x_n},$$

where P_k *is obtained from* $L_1, \ldots, L_{k-1}$ *by*

$$P_k = \Psi_1(L_{k-1}) + \Psi_2(Q_{k,2})_{\alpha_1=0} + \cdots + \Psi_n(Q_{k,n})_{\alpha_1=\alpha_2=\cdots=\alpha_{n-1}=0},$$

where for $2 \leq j \leq n$,

$$Q_{k,j} = \Phi_j(P_k) = a_{2,j}L_{k-2} + \cdots + a_{k-1,j}L_1.$$

and $a_{k,j}$ *are determined by solving* $L_k(F)_{\mathbf{x}=\hat{\mathbf{x}}} = 0$, *i.e.,*

$$\left[\frac{\partial F(\hat{\mathbf{x}})}{\partial x_2}, \ldots, \frac{\partial F(\hat{\mathbf{x}})}{\partial x_n}\right] \cdot [a_{k,2}, \ldots, a_{k,n}]^T = -P_k(F)_{\mathbf{x}=\hat{\mathbf{x}}}.$$

Assumptions on Approximate Solutions

Suppose we are given an approximate solution

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_{\mathbf{e}} + \hat{\mathbf{x}}_{\varepsilon},$$

where

- $\hat{\mathbf{x}}_{\mathbf{e}}$ denotes the exact singular solution of F with the multiplicity μ and $corank(F'(\hat{\mathbf{x}}_{\mathbf{e}})) = 1$.
- $\hat{\mathbf{x}}_{\varepsilon}$ denotes the error in the solution, $\|\hat{\mathbf{x}}_{\varepsilon}\| = \varepsilon \ll 1$.
- σ_i denotes the singular values of $F'(\hat{\mathbf{x}})$ satisfying

$$\sigma_i = \Theta(1),\ 1 \le i \le n-1, \quad \sigma_n = O(\varepsilon).$$

Regularized Newton Iterations

Theorem. *Under the assumptions, let* $A = F'(\hat{\mathbf{x}})$, *the solution* $\hat{\mathbf{y}}$ *of the following regularized least squares problem*

$$(A^*A + \sigma_n I_n)\hat{\mathbf{y}} = A^*\mathbf{b}$$

will satisfy

$$\|\hat{\mathbf{y}}\| = O(\varepsilon),\ \|F(\hat{\mathbf{x}} + \hat{\mathbf{y}})\| = O(\varepsilon^2),$$

where A^* *is the Hermitian transpose of* A, I_n *is the* $n \times n$ *identity matrix and* $\mathbf{b} = -F(\hat{\mathbf{x}})$.

Linear Transformations

- Compute the right singular vector $\mathbf{r}_1$ of $F'(\hat{\mathbf{x}}+\hat{\mathbf{y}})$ satisfying

$$\|F'(\hat{\mathbf{x}}+\hat{\mathbf{y}})\mathbf{r}_1\| = \sigma'_n = O(\varepsilon).$$

- Form a unitary matrix $R = [\mathbf{r}_1, \ldots, \mathbf{r}_n]$ and let

$$H(\mathbf{z}) = F(R\mathbf{z}).$$

  - $\hat{\mathbf{z}}_{\mathbf{e}} = R^{-1}\hat{\mathbf{x}}_{\mathbf{e}}$ is an exact root of H.
  - $\|H(\hat{\mathbf{z}})\| = \|F(\hat{\mathbf{x}}+\hat{\mathbf{y}})\| = O(\varepsilon^2)$,
  - $\left\| \frac{\partial H(\hat{\mathbf{z}})}{\partial z_1} \right\| = \|F'(\hat{\mathbf{x}}+\hat{\mathbf{y}})\mathbf{r}_1\| = \sigma'_n = O(\varepsilon).$

Consequence of the Regularized Newton Iteration

Theorem. *Under the same assumptions, after running one regularized Newton iteration, we have*

$$|\hat{z}_{1,\mathbf{e}} - \hat{z}_1| = \Theta(\varepsilon),$$

and

$$|\hat{z}_{i,\mathbf{e}} - \hat{z}_i| = O(\varepsilon^2), \text{ for } i = 2, \ldots, n.$$

Theorem. *Under the same assumptions, if the multiplicity* $\mu \geq 2$*, then we have*

$$\|L_i(H)_{\mathbf{z}=\hat{\mathbf{z}}}\| = O(\varepsilon^2), \text{ for } i = 0, \ldots, \mu - 2.$$

An Augmented Polynomial System

Theorem. *Under the same assumptions,* $\forall \mathbf{h} \in \mathbb{C}^n$ *satisfying* $\mathbf{h}^*\mathbf{r}_1 \neq 0$, $F'(\hat{\mathbf{x}}_{\mathbf{e}})\mathbf{r}_1 = 0$, *the augmented polynomial system*

$$J(\mathbf{x},\nu) := \begin{cases} F(\mathbf{x}), \\ F'(\mathbf{x})\cdot\nu, \\ \mathbf{h}^*\nu - 1, \end{cases}$$

has $(\hat{\mathbf{x}}_{\mathbf{e}}, \frac{\mathbf{r}_1}{\mathbf{h}^*\mathbf{r}_1})$ *as an isolated singular solution with the multiplicity* $\mu - 1$.

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Ojika et al.'82, Leykin et al'05: The multiplicity drops at least one.

Dayton, Zeng'05 Conjecture: The multiplicity drops exact one.

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- Compute the null vector $\mathbf{r}_1$ of $F'(\hat{\mathbf{x}}+\hat{\mathbf{y}})$, form a unitary matrix $R=[\mathbf{r}_1,\ldots,\mathbf{r}_n]$, and set

$$H(\mathbf{z}) := F(R\mathbf{z}), \hat{\mathbf{z}} := R^{-1}(\hat{\mathbf{x}}+\hat{\mathbf{y}}).$$

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- Solve

$$\left[P_\mu(H)_{\mathbf{z}=\hat{\mathbf{z}}},\frac{\partial H(\hat{\mathbf{z}})}{\partial z_2},\ldots,\frac{\partial H(\hat{\mathbf{z}})}{\partial z_s}\right]\mathbf{v} = -L_{\mu-1}(H)_{\mathbf{z}=\hat{\mathbf{z}}}$$

to obtain $\mathbf{v}$ and set $\delta := \frac{v_1}{\mu}$.

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  to obtain $\mathbf{v}$ and set $\delta := \frac{v_1}{\mu}$.
- Return $\hat{\mathbf{x}} := \hat{\mathbf{x}} + \hat{\mathbf{y}} + \delta \mathbf{r}_1$.

Rate of Convergence

Theorem. *Suppose* $\|\hat{\mathbf{x}}_\varepsilon\| = O(\varepsilon) \ll 1$, $\|F(\hat{\mathbf{x}})\| = O(\varepsilon)$ *and* $F'(\hat{\mathbf{x}})$ *has corank one approximately. Then the solution* $\hat{\mathbf{x}}$ *computed by our algorithm satisfies*

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Our refining algorithm converges **quadratically**.

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Remark: The biggest size of matrices appeared in our approach is $n \times n$, much smaller than $\mu n \times \mu n$ in [Dayton and Zeng'05] and $2^{\mu-1}n \times 2^{\mu-1}n$ [Ojika et al., Leykin et al.].

Example ([Ojika 1987] continued)

Given $\hat{\mathbf{x}} = (1.001, 2.002)$ and $F(\hat{\mathbf{x}}) \approx (0.4 \times 10^{-2}, 0.2 \times 10^{-2})$.

- Apply one regularized Newton iteration to F and $\hat{\mathbf{x}}$ to get

$$\hat{\mathbf{x}} + \hat{\mathbf{y}} \approx (0.99940, 2.0011998)$$
$$F(\hat{\mathbf{x}} + \hat{\mathbf{y}}) \approx (0.2588 \times 10^{-5}, 0.1295 \times 10^{-5});$$

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- Perform the linear transformation to get

$$\hat{\mathbf{z}}_{\varepsilon} \approx (0.1341 \times 10^{-2}, 0.1319 \times 10^{-5}); \quad \left\| \frac{\partial H(\hat{\mathbf{z}})}{\partial z_1} \right\| \approx 0.3 \times 10^{-6}.$$

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- Let $L_0 = D(0,0)$, $L_1 = D(1,0)$, and compute

$$L_2 = -0.9960D(2,0) + 0.0892D(0,1),$$
$$L_2(H)_{\mathbf{z}=\hat{\mathbf{z}}} = [-0.7168 \times 10^{-4}, 0.1433 \times 10^{-3}]^T.$$

Example ([Ojika 1987] continued)

- Form the matrix: $K = \begin{bmatrix} -0.0714 & 2.2350 \\ 0.0089 & 1.1182 \end{bmatrix}$. Solve the linear system $K\,\mathbf{v} = -L_2(H)_{\mathbf{z}=\hat{\mathbf{z}}}$ to obtain

$$\mathbf{v} = [-0.4014 \times 10^{-2}, -0.9611 \times 10^{-4}]^T.$$

The refined solution is

$$\hat{\mathbf{x}} = (1 - 0.14 \times 10^{-6}, 2 + 0.3 \times 10^{-5}).$$

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- After running the algorithm again, we get

$$\hat{\mathbf{x}} = (1 - 0.18 \times 10^{-11}, 2 + 0.7 \times 10^{-11}).$$

Algorithm Performance in Maple Digits = 14

System	Zero	t	s	μ	# Digits
Ojika1	$(1,2)$	2	2	3	$2 \rightarrow 5 \rightarrow 11 \rightarrow 15$
Ojika2	$(1,0,0)$	3	3	2	$2 \rightarrow 5 \rightarrow 10 \rightarrow 14$
Ojika3	$(-2.5,2.5,1)$	3	3	2	$2 \rightarrow 4 \rightarrow 9 \rightarrow 14$
Ojika4	$(0,0,10)$	3	3	3	$2 \rightarrow 3 \rightarrow 7 \rightarrow 13$
Decker2	$(0,0)$	2	2	4	$2 \rightarrow 5 \rightarrow 15$
DLZ	$(0,0)$	2	2	10	$2 \rightarrow 5 \rightarrow 16$
DZ3	$(\frac{2\sqrt{7}}{5}+\frac{\sqrt{5}}{5}, -\frac{\sqrt{7}}{5}+\frac{2\sqrt{5}}{5})$	2	2	5	$2 \rightarrow 5 \rightarrow 13$
Dayton2	$(0,0,0)$	3	3	5	$2 \rightarrow 3 \rightarrow 7 \rightarrow 13$
SY5	$(1,1)$	2	2	2	$2 \rightarrow 5 \rightarrow 11 \rightarrow 14$
Menzel1	$(1,1)$	3	2	2	$2 \rightarrow 5 \rightarrow 10 \rightarrow 14$

THANK YOU!