Inner interval addition/subtraction — algebraic properties

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Motivation: IEEE Interval Standard WG

Our talk developed from a motion presented by the same authors at the IEEE Interval Standard Working Group, see e. g.

Edmonson, W., Melquiond, G., IEEE Interval Standard Working Group - P1788: Current Status, Proc. 19th IEEE Symposium on Computer Arithmetic, 2009. ARITH 2009, 231–234.

IEEE interval arithmetic standard WG forum:

`http://grouper.ieee.org/groups/1788`

Motivation: Motion 12

Motion 12: Inner addition and subtraction over intervals

Inner addition and inner subtraction over $A = [\underline{a},\ \overline{a}]$, $B = [\underline{b},\ \overline{b}] \in \mathbb{IR}$:

$$A +^{-} B = \begin{cases} [\underline{a} + \overline{b},\ \overline{a} + \underline{b}] & \text{if } w(A) \geq w(B) \\ [\overline{a} + \underline{b},\ \underline{a} + \overline{b}] & \text{otherwise} \end{cases}$$

$$A -^{-} B = \begin{cases} [\underline{a} - \underline{b},\ \overline{a} - \overline{b}] & \text{if } w(A) \geq w(B) \\ [\overline{a} - \overline{b},\ \underline{a} - \underline{b}] & \text{otherwise} \end{cases}$$

$w(A) = \overline{a} - \underline{a}$

$[\underline{a} - \underline{b},\ \overline{a} - \overline{b}]$ for $w(A) \geq w(B)$ Hukuhara difference

Motivation: Motion 12 passes

29-Apr-2010

Motion P1788/M0012.01: Inner Addition And Subtraction

PASSES: 28-Apr-2010. Yes - 39; No - 0; of 70 registered

Needed for quorum: 36

George Corliss

Voting Tabulator

Motivation from Fuzzy Sets Theory

Recent papers on "Generalized Hukuhara difference" = inner subtraction

Bede B., L. Stefanini, Numerical solution of interval differential equations with generalized Hukuhara differentiability, Proc. IFSA-EUSFLAT 2009, Eds. J. P. Carvalho, et al., 730–735. http://www.eusflat.org/publications/proceedings/IFSA-EUSFLAT_2009/

Stefanini, L., New Tools in Fuzzy Arithmetic with Fuzzy Numbers, in: E. Hüllermeier, R. Kruse, and F. Hoffmann (Eds.): IPMU 2010, Part II, CCIS 81, 471–480, Springer, 2010.

Chalco-Cano, Y., L. Stefanini, H. Roman-Flores, Pi-difference of intervals and Pi-differentiability of interval-valued functions, submitted to Elsevier.

Purpose

The purpose of this talk is to:

— present interval arithmetic from a nonstandard perspective

— show that inner operations are natural and simple

— study their algebraic properties in a simple manner

What is new: algebraic properties are sudied within a focus on

i) inner addition instead on inner subtraction;

ii) errors (radii) instead on whole intervals

Intervals on the real line

$\mathbb{R}$ the set of real numbers, $(\mathbb{R}, \leq)$, $\mathbb{R}^+ = \{a \in \mathbb{R} \mid a \geq 0\}$

1. given $\underline{a}$, $\overline{a} \in \mathbb{R}$, $\underline{a} \leq \overline{a}$, define $\{x \mid \underline{a} \leq x \leq \overline{a}\}$

2. given $a' \in \mathbb{R}$, $a'' \in \mathbb{R}^+$ define

$\{x \mid -a'' \leq x - a' \leq a''\} = \{x \mid |x - a'| \leq a''\}$

Case 1. inf-sup notation $[\underline{a}, \overline{a}]$; Case 2. mid-rad $(a'; a'')$

$$\begin{aligned}
(a'; a'') + (b'; b'') &= (a' + b'; a'' + b''), \\
(a'; a'') - (b'; b'') &= (a' - b'; a'' + b''), \\
(a'; a'') +^- (b'; b'') &= (a' + b'; |a'' - b''|), \\
(a'; a'') -^- (b'; b'') &= (a' - b'; |a'' - b''|).
\end{aligned}$$

Real numbers

$\mathbb{R}$ the set of real numbers $\quad (\mathbb{R}, \leq)$

$\mathbb{R}^+ = \{a \in \mathbb{R} \mid a \geq 0\}, \quad \Lambda = \{+, -\}$

$a = (A; \alpha) \in \mathbb{R}^+ \otimes \Lambda$

$\mathbb{R}^+ \otimes \Lambda = \{(X; \xi) \mid X \in \mathbb{R}^+,\ \xi \in \Lambda\}$

Introduce addition in $\mathbb{R}$ via addition in $\mathbb{R}^+$

Addition of real numbers

Define "+" in $\mathbb{R}$ in terms of "+" in $\mathbb{R}^+$

so that $(\mathbb{R}, +) \cong (\mathbb{R}^+ \otimes \Lambda, +)$

$a = (A; \alpha),\ b = (B; \beta) \in \mathbb{R}$

Case $\alpha = \beta$: easy $\quad (A; \alpha) + (B; \beta) = (A + B; \alpha)$

Case $\alpha \neq \beta$: from elementary school textbooks

"Adding Real Numbers with Opposite Signs. Step 1: Take the difference of the absolute values. Step 2: Attach the sign of the number that has the larger absolute value, etc."

Addition of real numbers

$A +^{-} B = |A - B|$ the distance between A, B

$$A +^{-} B = \begin{cases} Y|_{B+Y=A} & \text{if } B \le A \\ X|_{A+X=B} & \text{if } B > A \end{cases}$$

$$\gamma(a,b) = \gamma((A;\alpha),(B;\beta)) = \begin{cases} \alpha & \text{if } B \le A \\ \beta & \text{if } B > A \end{cases}$$

General formula

$$a + b = (A; \alpha) + (B; \beta) = \begin{cases} (A + B; \alpha) & \text{if} \quad \alpha = \beta; \\ (A +^{-} B; \gamma(a, b)) & \text{if} \quad \alpha \neq \beta, \end{cases}$$

$$(A; \alpha) + (B; \beta) = (A +^{\alpha\beta} B; \gamma), \quad \gamma = \gamma(a, b).$$

$+^{+} = +, \quad +^{+-} = +^{-}$

Order relation preceding $\leq$

Similarly: Formulate "$\leq$" in $\mathbb{R}$, so that $(\mathbb{R}, \leq) \cong (\mathbb{R}^+ \otimes \Lambda, \leq)$

$$(A;\alpha) \leq (B;\beta) \Longleftrightarrow \begin{cases} (\alpha = -)\ \&\ (\beta = +) \text{ or} \\ (\alpha = \beta = -)\ \&\ (A \geq B) \text{ or} \\ (\alpha = \beta = +)\ \&\ (A \leq B). \end{cases}$$

$-5 < -2$ the real line model

Algebraic properties of $(\mathbb{R}, +)$

$(\mathbb{R}, +)$ is an additive group, that is

i) "+" is a closed (total) operation

ii) "+" is associative: $(a + b) + c = a + (b + c)$

iii) there is an identity 0, such that $a + 0 = a$ for all a

iv) for all a there exists an inverse element $-a$, such that $a + (-a) = 0$

Note: Property iv) induces subtraction $a - b = a + (-b)$

Algebraic properties of $(\mathbb{R}, +)$

$(\mathbb{R}, +) = (\mathbb{R}, +, 0, -, \leq)$

— property i) defines a magma

— properties i)–ii) define a semigroup

— properties i)–iii) define a monoid

— properties i)–iv) define a group

a group always obeys also:

— v) “cancellation law”: $a + x = b + x \Longrightarrow a = b$

Algebraic properties of $(\mathbb{R}, +)$

$(\mathbb{R}, +)$ satisfies also

— vi) commutative law $a + b = b + a$

$(\mathbb{R}, +)$ satisfies i)– vi) and is a commutative (abelian) group

$(\mathbb{R}, +)$ satisfies also

— vii) subtractability: for all $a, b \in \mathbb{R}$ there exists x such that $a + x = b$

Algebraic properties of $(\mathbb{R}^+, +, \leq)$

i)–iii) and vi) are satisfied so $(\mathbb{R}^+, +)$ is a commutative monoid

iv) fails; equation $A + X = 0$ has no solution when $A > 0$

v) $A + X = B + X \Longrightarrow A = B$ holds true

a cancellative monoid may not be a group

Subtractability vii) does not hold as $A + X = B$ does not possess a solution in general

Algebraic properties of $(\mathbb{R}^+, +, \leq)$

$A + X = B$ possesses a solution when $A \leq B$

This solution can be expressed in terms of operation "$+^-$"

$$A +^- B = \begin{cases} Y|_{B+Y=A} & \text{if } B \leq A; \\ X|_{A+X=B} & \text{if } B > A. \end{cases}$$

the solution X of equation $A + X = B$ when $A \leq B$ is $X = A +^- B$

c-addition "$+^-$" appears naturally in $(\mathbb{R}^+, +)$

We next study c-addition "$+^-$" in some detail

Algebraic properties of $(\mathbb{R}^+, +^-)$

— i) "$+^-$" is a closed; $(\mathbb{R}^+, +^-)$ is a magma

— associativity ii) $(A +^- B) +^- C = A +^- (B +^- C)$ fails,
$(7 +^- 5) +^- 3 \neq 7 +^- (5 +^- 3)$

— iii) existence of identity $A +^- 0 = A$ for all $A \in \mathbb{R}^+$, holds true;
$(\mathbb{R}^+, +^-)$ is an unital magma

— iv) existence of an inverse holds true as well $A +^- A = 0$ for all A

— v) "cancellation law": $A +^- X = B +^- X \Longrightarrow A = B$ fails,
$1 +^- 3 = 5 +^- 3$, but $1 \neq 3$

— vi) commutative law $A +^- B = B +^- A$ holds true

Algebraic properties of $(\mathbb{R}^+, +^-)$

System $(\mathbb{R}^+, +^-)$ satisfies properties i), iii) and vi) so it is commutative unital magma

c-associativity Let $A, B, C \in \mathbb{R}^+$ be such that $B \geq A, B \geq C$; then $(A +^- B) +^- C = A +^- (B +^- C)$

c-cancellation Let $A, B \in \mathbb{R}^+$. Equation $A +^- X = B +^- X$ is satisfied for $X = (A + B)/2$.

If $X \neq (A + B)/2$, then $A +^- X = B +^- X \Longrightarrow A = B$.

The system $(\mathbb{R}^+, +, 0, +^-, \leq)$

We focus on the (extended) monoid $(\mathbb{R}^+, +) = (\mathbb{R}^+, +, 0, +^- \leq)$

C-subtractability

i) For $A, B \in \mathbb{R}^+$, such that $A \leq B$, the unique solution of $A + X = B$ is $X = B +^- A$

ii) Equation $A +^- X = B$ has a solution $X = A + B$ for $A, B \in \mathbb{R}^+$. If $A, B \in \mathbb{R}^+$ are such that $A \geq B > 0$, then equation $A +^- X = B$ has one more solution $X = A +^- B$

The system $(\mathbb{R}^+, +, 0, +^-, \leq)$

C-associativity $(a + b) + c = a + (b + c), \quad a, b, c \in \mathbb{R}$

$((A; \alpha) + (B; \beta)) + (C; \gamma) = (A; \alpha) + ((B; \beta) + (C; \gamma))$

$$((A +^{\alpha\beta} B; \mu(a, b))) + (C; \gamma) = (A; \alpha) + ((B +^{\beta\gamma} C; \mu(b, c))) \qquad (1)$$

$$\phi(A, B) = \begin{cases} +, & \text{if } A \geq B; \\ -, & \text{otherwise.} \end{cases}$$

$$(A + B) +^- C = A +^{\phi(B,C)} (B +^- C), \quad (\alpha = \beta = -\gamma)$$

The system $(\mathbb{R}^+, +, 0, +^-, \leq)$

C-associativity

$$
\begin{aligned}
(A + B) +^- C &= A +^{\phi(B,C)} (B +^- C);\\
(A +^- B) + C &= \begin{cases} A +^{-\phi(B,C)} (B +^- C), & A \geq B,\\ A +^- (B + C), & A < B; \end{cases}\\
(A +^- B) +^- C &= \begin{cases} A +^{-\phi(B,C)} (B +^- C), & A < B,\\ A +^- (B + C), & A \geq B. \end{cases}
\end{aligned}
$$

For each triple $A, B, C \in \mathbb{R}^+$ and each pair $\theta_1, \theta_2 \in \{+, -\}$, there exist a pair $\theta_3, \theta_4 \in \{+, -\}$, such that

$$(A +^{\theta_1} B) +^{\theta_2} C = A +^{\theta_3} (B +^{\theta_4} C)$$

The system $(\mathbb{R}^+, +, 0, +^-, \leq)$

C-associativity four elements A, B, C, D

$\gamma = \phi(A,C)\phi(B,D), \quad \delta = \phi(A,B)\phi(C,D)$

$$
\begin{aligned}
(A+B) +^- (C+D) &= (A +^- C) +^{\gamma} (B +^- D);\\
(A +^- B) + (C +^- D) &= \begin{cases} (A +^- C) +^{-\gamma} (B +^- D), & \text{if } \delta < 0;\\ (A+C) +^- (B+D), & \text{if } \delta \geq 0; \end{cases}\\
(A +^- B) +^- (C +^- D) &= \begin{cases} (A +^- C) +^{-\gamma} (B +^- D), & \text{if } \delta \geq 0;\\ (A+C) +^- (B+D), & \text{if } \delta < 0. \end{cases}
\end{aligned}
$$

The system $(\mathbb{R}^+, +, 0, +^-, \leq)$

C-associativity

Example. $A \in [3,4],\ B \in [1,2],\ C \in [4,5],\ D \in [6,7]$

Since $\gamma = -, \delta = +$, we have

$$\begin{aligned} (A + B) +^- (C + D) &= (A +^- C) +^- (B +^- D) \\ (A +^- B) + (C +^- D) &= (A + C) +^- (B + D) \\ (A +^- B) +^- (C +^- D) &= (A +^- C) + (B +^- D) \end{aligned}$$

Note in the last relation

$$(A +^- B) +^- (C +^- D) = (A +^- C) +^{-\gamma} (B +^- D) \text{ if } \delta \geq 0,$$

the condition $\delta \geq 0$ is not as restrictive as it looks like, due to commutativity of "$+^-$"

Order isotonicity

The preceding order "$\leq$" is consistent with both addition and c-addition.

For $X, X_1, C \in \mathbb{R}^+$ $X \leq X_1 \implies X + C \leq X_1 + C$

As a consequence $X \leq X_1$, $Y \leq Y_1 \implies X + Y \leq X_1 + Y_1$

Inverse isotonicity of addition. If $A, B, C \in \mathbb{R}^+$, then

$$C + A \leq C + B \Longrightarrow A \leq B$$

in particular $C + A = C + B \Longrightarrow A = B$ (cancellation law)

Order isotonicity

C-inclusion isotonicity w.r.t. "$+^-$"

Let $X, X_1, Y, Y_1 \in \mathbb{R}^+$. Assuming $X \geq X_1, Y \leq Y_1$, we have

$$\text{if } X \leq Y, \text{ then } X +^- Y \leq X_1 +^- Y_1,$$
$$\text{if } X_1 \geq Y_1, \text{ then } X +^- Y \geq X_1 +^- Y_1$$

In the special case $Y = Y_1 = C$ we have:

For $X, X_1, C \in \mathbb{R}^+$ $X \geq X_1$ implies

$$\text{if } X \leq C, \text{ then } X +^- C \leq X_1 +^- C,$$
$$\text{if } X_1 \geq C, \text{ then } X +^- C \geq X_1 +^- C.$$

The system $(\mathbb{R}^+, +, 0, +^-, \leq)$

Axiom/System	$(\mathbb{R}, +)$	$(\mathbb{R}^+, +)$	$(\mathbb{R}^+, +^-)$	$(\mathbb{R}^+, +, +^-)$
Closure	Yes	Yes	Yes	Yes & Yes
Associativity	Yes	Yes	C	Yes & C
Identity	Yes	Yes	Yes	Yes & Yes
Inverse	Yes	No	Yes	C & Yes
Cancellation	Yes	Yes	C	Yes & C
Commutativity	Yes	Yes	Yes	Yes & Yes
Subtractability	Yes	No	No	C & C

Approximate numbers

Approximate numbers — intervals in MR-form

$a = (a'; a'')$, $a' \in \mathbb{R}$ (midpoint),

$a'' \in \mathbb{R}^+$ (radius, error bound)

$a = (a'; a'') \in \mathbb{R} \otimes \mathbb{R}^+$.

The binary arithmetic operations in $(\mathbb{R}, +, \leq)$ are addition "$a + b$" and subtraction "$a - b$"

whereas in $(\mathbb{R}^+, +, +^-, \leq)$ are addition "$A + B$" and inner addition "$A +^- B$"

Approximate numbers: operations

$$
\begin{aligned}
(a'; a'') + (b'; b'') &= (a' + b'; a'' + b''),\\
(a'; a'') +^{-} (b'; b'') &= (a' + b'; a'' +^{-} b''),\\
(a'; a'') \neg (b'; b'') &= (a' - b'; a'' + b''),\\
(a'; a'') -^{-} (b'; b'') &= (a' - b'; a'' +^{-} b'').
\end{aligned}
$$

Approximate numbers: operations

$$
\begin{aligned}
(a'; a'') + (b'; b'') &= (a' + b'; a'' + b''),\\
(a'; a'') +^{-} (b'; b'') &= (a' + b'; |a'' - b''|),\\
(a'; a'') \neg (b'; b'') &= (a' - b'; a'' + b''),\\
(a'; a'') -^{-} (b'; b'') &= (a' - b'; |a'' - b''|).
\end{aligned}
$$

Hints for applications

$$\exp(X) = 1 + X/1! + X^2/2! + X^3/3! + X^4/4! + ..., \quad X \geq 0$$

$$\exp(X) = 1 + X/1! +^- X^2/2! + X^3/3! +^- X^4/4! + ..., \quad -1 \leq X < 0$$

$$\ln(1 + X) = X - X^2/2 + X^3/3 - X^4/4 + ..., \quad 0 < X \leq 1$$

$$\ln(1 + X) = X -^- X^2/2 + X^3/3 -^- X^4/4 + ..., \quad -1 < X \leq 0$$

Order of the execution — from left to right

Ranges $X^n = \{x^n | x \in X\}$ assumed exact

Conclusions

Our approach to IA shows

i) inner operations are natural

ii) mid-rad is a natural form