“Spanish version” of formal approach for enclosing solution sets to interval linear systems

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Interval linear systems of equations

$$\left\{\begin{array}{ccccccccc} \boldsymbol{a}_{11}x_1 & + & \boldsymbol{a}_{12}x_2 & + & \ldots & + & \boldsymbol{a}_{1n}x_n & = & \boldsymbol{b}_1, \\ \boldsymbol{a}_{21}x_1 & + & \boldsymbol{a}_{22}x_2 & + & \ldots & + & \boldsymbol{a}_{2n}x_n & = & \boldsymbol{b}_2, \\ & \vdots & & & \ddots & & & \vdots & \\ \boldsymbol{a}_{n1}x_1 & + & \boldsymbol{a}_{n2}x_2 & + & \ldots & + & \boldsymbol{a}_{nn}x_n & = & \boldsymbol{b}_n, \end{array}\right.$$

or, briefly,

$$\boldsymbol{A}x = \boldsymbol{b}$$

with an interval matrix $\boldsymbol{A} = (\,\boldsymbol{a}_{ij})$ and an interval vector $\boldsymbol{b} = (\,\boldsymbol{b}_i)$.

Interval linear systems of equations

$$\boldsymbol{A}x = \boldsymbol{b}$$

— a family of point linear systems $Ax = b$ with $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$.

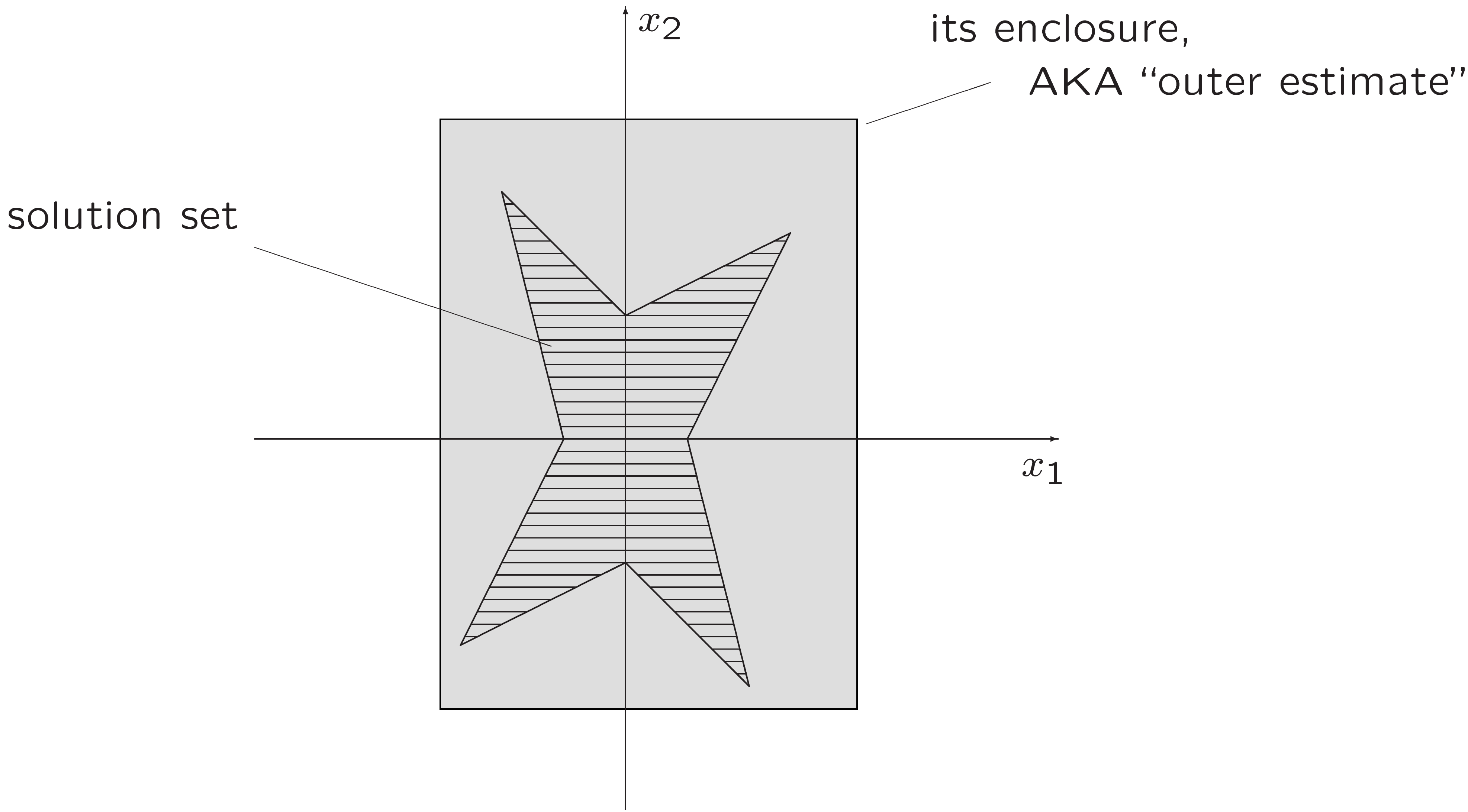
Solution set

of the interval linear system of equations —

$$\varXi(\boldsymbol{A}, \boldsymbol{b}) \;=\; \left\{\, x \in \mathbb{R}^n \mid (\exists\, A \in \boldsymbol{A})(\exists\, b \in \boldsymbol{b})(\, Ax = b\,) \,\right\}$$

Also *united solution set* . . .

“Outer problem”



Problem statement

$$\boldsymbol{A}x = \boldsymbol{b}$$

and the interval matrix $\boldsymbol{A}$ is supposed to be regular

Find (as tight as possible) a box $\boldsymbol{U}$,
that contains the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$
to the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$

Linear systems of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \qquad \vdots \qquad\qquad\qquad \ddots \qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n, \end{cases}$$

or, briefly,

$$Ax = b$$

with a matrix $A = (a_{ij})$ and a vector $b = (b_i)$.

Miranda theorem

Let $F : \mathbb{R}^n \to \mathbb{R}^n$, $F(x) = \big(F_1(x), F_2(x), \ldots, F_n(x)\big)^\top$ be a continuous function on the axis-aligned box $\boldsymbol{X} \subset \mathbb{R}^n$ and there holds

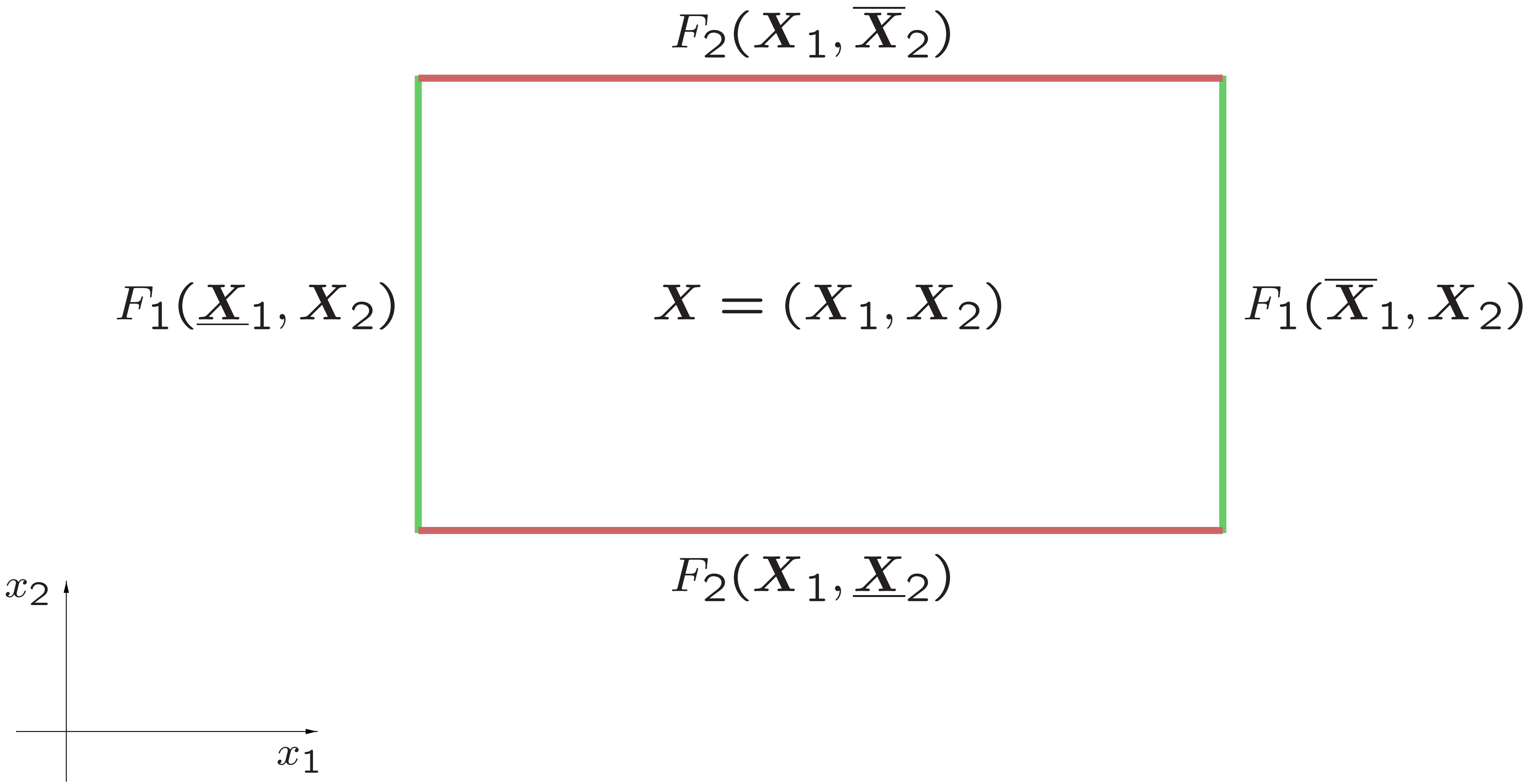
$$F_i\big(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{i-1}, \underline{\boldsymbol{X}}_i, \boldsymbol{X}_{i+1}, \ldots, \boldsymbol{X}_n\big) \times F_i\big(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{i-1}, \overline{\boldsymbol{X}}_i, \boldsymbol{X}_{i+1}, \ldots, \boldsymbol{X}_n\big) \leq 0$$

for each $i = 1, 2, \ldots, n$, that is, the ranges of values of the components $F_i(x)$ over the corresponding opposite faces of the box $\boldsymbol{X}$ have different signs.

Then the box $\boldsymbol{X}$ contains a zero of the function F,

i. e. a point $\tilde{x}$ where $F(\tilde{x}) = 0$.

Miranda theorem



Enclosing solutions to linear systems

Given $F(x) = Ax - b$ with $n \times n$-matrix $A = (a_{ij})$ and n-vector $b = (b_i)$, Miranda theorem implies:

If an interval box $\boldsymbol{X}$ satisfies, for each $i = 1, 2, \ldots, n$, the inequalities

$$a_{ii}\underline{\boldsymbol{X}}_i + \sum_{j \neq i} a_{ij}\boldsymbol{X}_j - b_i \leq 0 \quad \text{and} \quad a_{ii}\overline{\boldsymbol{X}}_i + \sum_{j \neq i} a_{ij}\boldsymbol{X}_j - b_i \geq 0 \qquad (\blacktriangle)$$

or

$$a_{ii}\underline{\boldsymbol{X}}_i + \sum_{j \neq i} a_{ij}\boldsymbol{X}_j - b_i \geq 0 \quad \text{and} \quad a_{ii}\overline{\boldsymbol{X}}_i + \sum_{j \neq i} a_{ij}\boldsymbol{X}_j - b_i \leq 0, \qquad (\blacktriangledown)$$

where all the operations are performed in interval arithmetic,
then $\boldsymbol{X}$ contains the solution of the linear system $Ax = b$.

. . . the inequalities ($\blacktriangle$) hold for $a_{ii} \geq 0$,
while ($\blacktriangledown$) is true for $a_{ii} \leq 0$.

Kaucher complete interval arithmetic $\mathbb{KR}$

Elements of $\mathbb{KR}$ are pairs $\boldsymbol{x} := [\underline{\boldsymbol{x}}, \overline{\boldsymbol{x}}]$, and the relation $\underline{\boldsymbol{x}} \leq \overline{\boldsymbol{x}}$ is not obligatory

$$\begin{cases} \textit{proper intervals } \boldsymbol{x}, & \text{with } \underline{\boldsymbol{x}} \leq \overline{\boldsymbol{x}} \\ \textit{improper intervals } \boldsymbol{x}, & \text{with } \underline{\boldsymbol{x}} > \overline{\boldsymbol{x}} \end{cases}$$

Dualization —

$$\text{dual}\,[\underline{\boldsymbol{x}}, \overline{\boldsymbol{x}}] := [\overline{\boldsymbol{x}}, \underline{\boldsymbol{x}}]$$

"Inclusion" ordering

$$\boldsymbol{x} \subseteq \boldsymbol{y} \quad \Longleftrightarrow \quad \underline{\boldsymbol{x}} \geq \underline{\boldsymbol{y}} \quad \text{and} \quad \overline{\boldsymbol{x}} \leq \overline{\boldsymbol{y}}$$

Kaucher complete interval arithmetic $\mathbb{KR}$

Addition and multiplication by a number

$$\boldsymbol{x} + \boldsymbol{y} \;:=\; \left[\underline{\boldsymbol{x}} + \underline{\boldsymbol{y}}, \overline{\boldsymbol{x}} + \overline{\boldsymbol{y}}\right]$$

$$\lambda \cdot \boldsymbol{x} \;:=\; \begin{cases} [\,\lambda\,\underline{\boldsymbol{x}}, \lambda\,\overline{\boldsymbol{x}}\,], & \text{if } \lambda \geq 0 \\ [\,\lambda\,\overline{\boldsymbol{x}}, \lambda\,\underline{\boldsymbol{x}}\,], & \text{otherwise} \end{cases}$$

Every $\boldsymbol{x} \in \mathbb{KR}$ has a unique opposite $\operatorname{opp} \boldsymbol{x}$:

$$\boldsymbol{x} + \operatorname{opp} \boldsymbol{x} = 0 \quad \Rightarrow \quad \operatorname{opp} \boldsymbol{x} := [\,-\underline{\boldsymbol{x}}, -\overline{\boldsymbol{x}}\,]$$

We denote

$$\boldsymbol{x} \ominus \boldsymbol{y} \;:=\; \boldsymbol{x} + \operatorname{opp} \boldsymbol{y}$$

Kaucher complete interval arithmetic $\mathbb{KR}$

$\mathcal{P} := \{\, \boldsymbol{x} \in \mathbb{KR} \mid (\underline{\boldsymbol{x}} \geq 0)\,\&\,(\overline{\boldsymbol{x}} \geq 0) \,\}$ $\mathcal{Z} := \{\, \boldsymbol{x} \in \mathbb{KR} \mid \underline{\boldsymbol{x}} \leq 0 \leq \overline{\boldsymbol{x}} \,\}$

$-\mathcal{P} := \{\, -\boldsymbol{x} \mid \boldsymbol{x} \in \mathcal{P} \,\}$ $\text{dual}\,\mathcal{Z} := \{\, \boldsymbol{x} \in \mathbb{KR} \mid \text{dual}\,\boldsymbol{x} \in \mathcal{Z} \,\}$

$\cdot$	$\boldsymbol{b} \in \mathcal{P}$	$\boldsymbol{b} \in \mathcal{Z}$	$\boldsymbol{b} \in -\mathcal{P}$	$\boldsymbol{b} \in \text{dual}\,\mathcal{Z}$
$\boldsymbol{a} \in \mathcal{P}$	$[\,\underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}\,]$	$[\,\overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}\,]$	$[\,\overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}, \underline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}\,]$	$[\,\underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}, \underline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}\,]$
$\boldsymbol{a} \in \mathcal{Z}$	$[\,\underline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}\,]$	$[\,\min\{\underline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}\}, \max\{\underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}\}\,]$	$[\,\overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}, \underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}\,]$	0
$\boldsymbol{a} \in -\mathcal{P}$	$[\,\underline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}\,]$	$[\,\underline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}, \underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}\,]$	$[\,\overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}, \underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}\,]$	$[\,\overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}\,]$
$\boldsymbol{a} \in \text{dual}\,\mathcal{Z}$	$[\,\underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}\,]$	0	$[\,\overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}, \underline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}\,]$	$[\,\max\{\underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}\}, \min\{\underline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}\}\,]$

Kaucher complete interval arithmetic $\mathbb{KR}$

Every $\boldsymbol{x} \in \mathbb{KR}$ with $\underline{\boldsymbol{x}}\,\overline{\boldsymbol{x}} > 0$ has a unique inverse $\operatorname{inv} \boldsymbol{x}$:

$$\boldsymbol{x} \cdot \operatorname{inv} \boldsymbol{x} = 1 \quad \Rightarrow \quad \operatorname{inv} \boldsymbol{x} := [\,1/\underline{\boldsymbol{x}}, 1/\overline{\boldsymbol{x}}\,]$$

Subtraction and division:

$$\boldsymbol{x} - \boldsymbol{y} = \boldsymbol{x} + (-1) \cdot \boldsymbol{y}$$

$$\boldsymbol{x} \,/\, \boldsymbol{y} = \boldsymbol{x} \cdot [\,1/\overline{\boldsymbol{y}}, 1/\underline{\boldsymbol{y}}\,] \qquad \text{for} \quad \underline{\boldsymbol{y}}\,\overline{\boldsymbol{y}} > 0$$

All the interval operations in $\mathbb{KR}$ are *inclusion monotonic*:

$$\boldsymbol{x} \subseteq \boldsymbol{x}',\ \boldsymbol{y} \subseteq \boldsymbol{y}' \quad \Rightarrow \quad \boldsymbol{x} \star \boldsymbol{y} \subseteq \boldsymbol{x}' \star \boldsymbol{y}' \qquad \text{for} \quad \star \in \{\,+, -, \cdot, /\,\}.$$

Interval vectors and matrices

Sum (difference) of two interval matrices is formed by sums (differences) of the corresponding elements of the operands.

For the matrices $\boldsymbol{X} = (\,\boldsymbol{x}_{ij})$ and $\boldsymbol{Y} = (\,\boldsymbol{y}_{ij})$, the product $\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{Z} = (\,\boldsymbol{z}_{ij})$ is defined so that

$$\boldsymbol{z}_{ij} = \sum_{k=1}^{l} \boldsymbol{x}_{ik}\boldsymbol{y}_{kj}.$$

Topology on interval spaces $\mathbb{IR}^n$ and $\mathbb{KR}^n$ is defined by the metric

$$\text{dist}\,(\boldsymbol{x}, \boldsymbol{y}) \;:=\; \max\{\|\underline{\boldsymbol{x}} - \underline{\boldsymbol{y}}\|, \|\overline{\boldsymbol{x}} - \overline{\boldsymbol{y}}\|\},$$

where $\|\cdot\|$ is a vector norm on $\mathbb{R}^n$, or by a multimetric

$$\text{Dist}\,(\boldsymbol{x}, \boldsymbol{y}) = \Big(\text{dist}\,(\boldsymbol{x}_1, \boldsymbol{y}_1), \ldots, \text{dist}\,(\boldsymbol{x}_n, \boldsymbol{y}_n)\Big)^{\top}$$

for vectors.

Enclosing solutions to linear systems

Let $a_{ii} \geq 0$, i.e. the relations (▲) are true. Then

$$\overline{\left(a_{ii}\underline{\boldsymbol{X}}_i + \sum_{j\neq i} a_{ij}\boldsymbol{X}_j\right)} - b_i \leq 0 \qquad \text{and} \qquad \underline{\left(a_{ii}\overline{\boldsymbol{X}}_i + \sum_{j\neq i} a_{ij}\boldsymbol{X}_j\right)} - b_i \geq 0,$$

or

$$\overline{\left(a_{ii}\cdot\overline{\text{dual }\boldsymbol{X}}_i + \sum_{j\neq i} a_{ij}\boldsymbol{X}_j\right)} - b_i \leq 0, \qquad \underline{\left(a_{ii}\cdot\underline{\text{dual }\boldsymbol{X}}_i + \sum_{j\neq i} a_{ij}\boldsymbol{X}_j\right)} - b_i \geq 0.$$

Since $a_{ii} \geq 0$, we have

$$a_{ii}\cdot\underline{\text{dual }\boldsymbol{X}}_i = \underline{a_{ii}\cdot\text{dual }\boldsymbol{X}_i} \qquad \text{and} \qquad a_{ii}\cdot\overline{\text{dual }\boldsymbol{X}}_i = \overline{a_{ii}\cdot\text{dual }\boldsymbol{X}_i},$$

and therefore

$$\overline{\left(a_{ii}\cdot\text{dual }\boldsymbol{X}_i + \sum_{j\neq i} a_{ij}\boldsymbol{X}_j\right)} - b_i \leq 0, \qquad \underline{\left(a_{ii}\cdot\text{dual }\boldsymbol{X}_i + \sum_{j\neq i} a_{ij}\boldsymbol{X}_j\right)} - b_i \geq 0.$$

Enclosing solutions to linear systems

Let $a_{ii} \le 0$, i.e. the relations (▼) are true. Then

$$\underline{\left(a_{ii}\underline{\boldsymbol{X}}_i + \sum_{j\neq i} a_{ij}\boldsymbol{X}_j \right)} - b_i \ge 0 \qquad \text{and} \qquad \overline{\left(a_{ii}\overline{\boldsymbol{X}}_i + \sum_{j\neq i} a_{ij}\boldsymbol{X}_j \right)} - b_i \le 0,$$

or

$$\underline{\left(a_{ii} \cdot \overline{\text{dual } \boldsymbol{X}}_i + \sum_{j\neq i} a_{ij}\boldsymbol{X}_j \right)} - b_i \ge 0, \qquad \overline{\left(a_{ii} \cdot \underline{\text{dual } \boldsymbol{X}}_i + \sum_{j\neq i} a_{ij}\boldsymbol{X}_j \right)} - b_i \le 0.$$

Since $a_{ii} \le 0$, we have

$$a_{ii} \cdot \overline{\text{dual } \boldsymbol{X}}_i = \underline{a_{ii} \cdot \text{dual } \boldsymbol{X}_i} \qquad \text{and} \qquad a_{ii} \cdot \underline{\text{dual } \boldsymbol{X}}_i = \overline{a_{ii} \cdot \text{dual } \boldsymbol{X}_i},$$

and therefore

$$\underline{\left(a_{ii} \cdot \text{dual } \boldsymbol{X}_i + \sum_{j\neq i} a_{ij}\boldsymbol{X}_j \right)} - b_i \ge 0, \qquad \overline{\left(a_{ii} \cdot \text{dual } \boldsymbol{X}_i + \sum_{j\neq i} a_{ij}\boldsymbol{X}_j \right)} - b_i \le 0.$$

Enclosing solutions to linear systems

The formulas for $a_{ii} \geq 0$ and $a_{ii} \leq 0$ coincide!

On the whole, regardless of the sign of a_{ii},
we have in the complete interval arithmetic $\mathbb{KR}$

Proposition 1

If the box $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n)^\top$ satisfy the conditions

$$a_{ii} \cdot \text{dual}\, \boldsymbol{X}_i + \sum_{j \neq i} a_{ij} \boldsymbol{X}_j - b_i \subseteq 0, \qquad i = 1, 2, \ldots, n,$$

then $\boldsymbol{X}$ contains the solution to the linear equations system $Ax = b$ with $n \times n$-matrix $A = (a_{ij})$ and n-vector $b = (b_i)$ in the right-hand side.

Enclosures for interval linear systems

$$\boldsymbol{A}x = \boldsymbol{b}$$

— since the interval linear system is a family of point systems $Ax = b$ with $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$, we arrive at

Proposition 2

Let $\boldsymbol{A} = (\boldsymbol{a}_{ij}) \in \mathbb{IR}^{n\times n}$ be an interval matrix, $\boldsymbol{b} \in \mathbb{IR}^n$ and $\boldsymbol{X} \in \mathbb{IR}^n$ be interval vectors. If, for each $i = 1, 2, \ldots, n$, there holds the inclusion

$$\boldsymbol{a}_{ii} \cdot \text{dual}\, \boldsymbol{X}_i + \sum_{j\neq i} \boldsymbol{a}_{ij}\boldsymbol{X}_j \ominus \boldsymbol{b}_i \ \subseteq\ 0,$$

then the box $\boldsymbol{X}$ contains the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ of the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$.

Definition

An interval vector $\boldsymbol{x} = (\,\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\,)^\top$ is called formal solution of the interval system of equations

$$\left\{\begin{array}{rcl} F_1(\,\boldsymbol{a}_1, \ldots, \boldsymbol{a}_l, x_1, \ldots, x_n) & = & \boldsymbol{b}_1, \\ F_2(\,\boldsymbol{a}_1, \ldots, \boldsymbol{a}_l, x_1, \ldots, x_n) & = & \boldsymbol{b}_2, \\ \vdots \qquad \ddots & & \vdots \\ F_m(\,\boldsymbol{a}_1, \ldots, \boldsymbol{a}_l, x_1, \ldots, x_n) & = & \boldsymbol{b}_m, \end{array}\right.$$

with $\boldsymbol{a}_1$, ..., $\boldsymbol{a}_l$, $\boldsymbol{b}_1$, ..., $\boldsymbol{b}_m \in \mathbb{IR}$, providing that it turns the equations of the system into equalities after substituting it instead of the unknown variable and carrying out all the operations according to the rules of an interval arithmetic.

Formal solutions

. . . is an object that corresponds to usual mathematical concept of the solution to an equation, altough considered in an unusual algebraic system, interval arithmetic $\mathbb{IR}$ or $\mathbb{KR}$ or any other.

S. Berti (1969)

K.L.E. Nickel (1982)

H. Ratschek, W. Sauer (1982) — *algebraic solutions*

Main result

Proposition 3

Let the mapping $\mathcal{S} : \mathbb{KR}^n \to \mathbb{KR}^n$, depending on the parameters $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ и $\boldsymbol{b} = (\boldsymbol{b}_i)$, be defined in the component-wise manner as

$$\mathcal{S}_i(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{x}) \;=\; \boldsymbol{a}_{ii} \cdot \operatorname{dual} \boldsymbol{x}_i \;+\; \sum_{j \neq i} \boldsymbol{a}_{ij} \boldsymbol{x}_j \;-\; \boldsymbol{b}_i, \qquad i = 1, 2, \ldots, n.$$

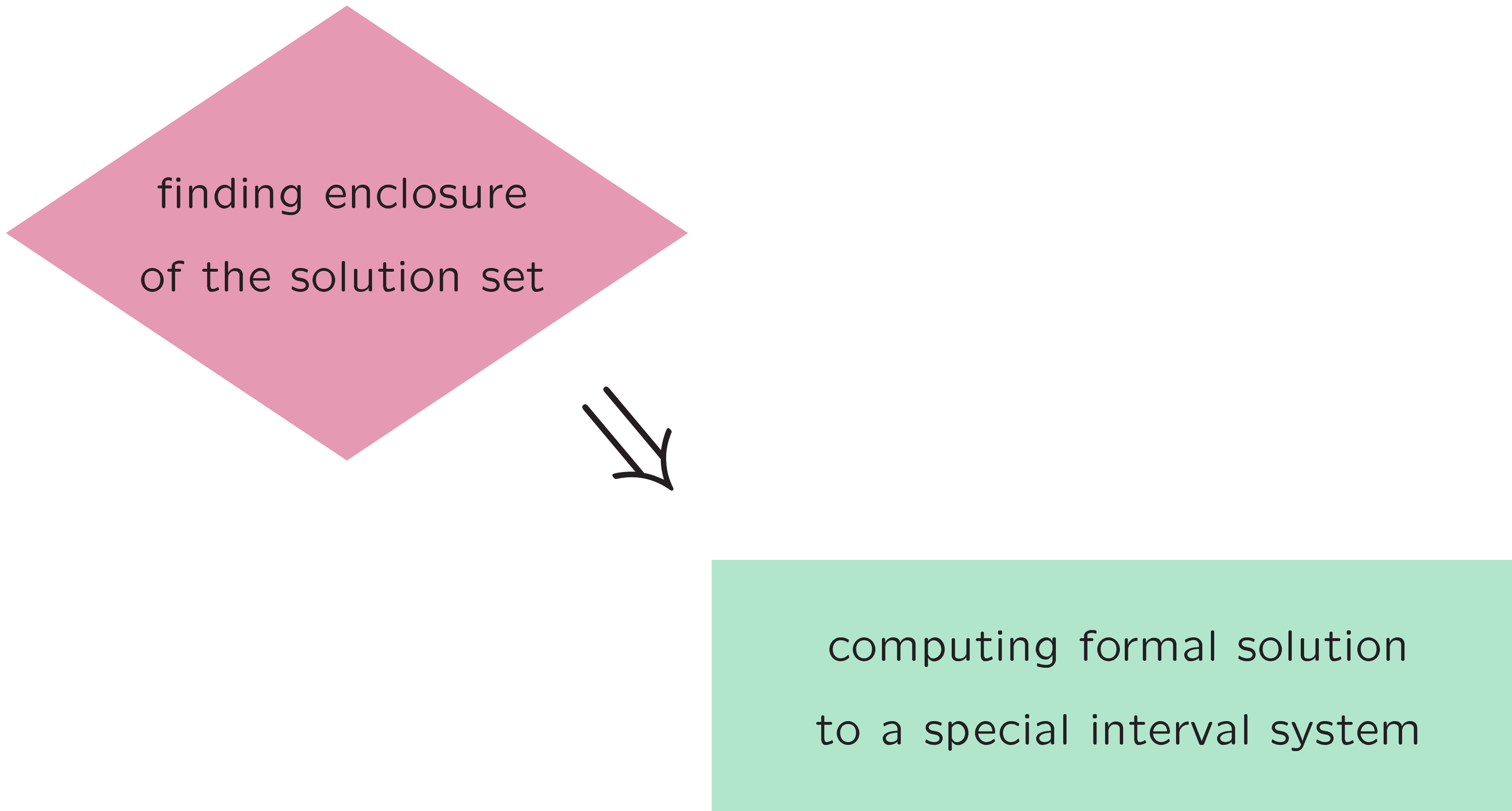
Proper formal solution to the interval equations system

$$\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, x) = 0$$

contains the solution set $\varXi(\boldsymbol{A}, \boldsymbol{b})$ of the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$.

Formal approach

— AKA “formal-algebraic” approach



Predecessors

M.A. Sainz, E. Gardeñes, L. Jorba
Formal solution to systems of interval linear or non-linear equations // *Reliable Computing*. – 2002. – Vol. 8. – P. 189–211.

M.A. Sainz, E. Gardeñes, L. Jorba
Interval estimations of solution sets to real-valued systems of linear or non-linear equations // *Reliable Computing*. – 2002. – Vol. 8. – P. 283–305.

Barcelona and Girona, Spain

“Spanish version” of formal approach

Proposition 3

Let the mapping $\mathcal{S} : \mathbb{KR}^n \to \mathbb{KR}^n$, depending on the parameters $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ и $\boldsymbol{b} = (\boldsymbol{b}_i)$, be defined in the component-wise manner as

$$\mathcal{S}_i(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{x}) \;=\; \boldsymbol{a}_{ii} \cdot \mathrm{dual}\, \boldsymbol{x}_i \;+\; \sum_{j \neq i} \boldsymbol{a}_{ij} \boldsymbol{x}_j \;-\; \boldsymbol{b}_i, \qquad i = 1, 2, \ldots, n.$$

Proper formal solution to the interval equations system

$$\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, x) = 0$$

contains the solution set $\varXi(\boldsymbol{A}, \boldsymbol{b})$ of the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$.

Natural questions

0) When does the proper formal solution exist for $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{x}) = 0$?

1) Correlation with the other existing techniques?

2) What is the enclosure quality?

3) How can we compute the formal solutions?

Traditional version of formal approach

The solution set to the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$ with $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and $\boldsymbol{b} \in \mathbb{IR}^{n}$ coincides with the solution set to the system

$$x = \boldsymbol{C}x + \boldsymbol{d},$$

where $\boldsymbol{C} = I - \Lambda\boldsymbol{A}$, $\boldsymbol{d} = \Lambda\boldsymbol{b}$, and Λ is a regular diagonal matrix.

Apostolatos-Kulisch theorem (modern formulation)

If the matrix $\boldsymbol{C} \in \mathbb{IR}^{n\times n}$ satisfies $\rho(|\boldsymbol{C}|) < 1$, then the interval linear system

$$x = \boldsymbol{C}x + \boldsymbol{d}$$

has a unique proper formal solution that contains its solution set.

Existence of proper formal solutions

For the formal solution of the equation $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{x}) = 0$ to be proper, all the diagonal elements of the matrix $\boldsymbol{A}$ must not contain zero.

Overall, $\boldsymbol{A}$ should be an interval H-matrix . . .

Let us represent the matrix of the initial system $\boldsymbol{A}x = \boldsymbol{b}$ as the sum of diagonal and off-diagonal parts

$$\boldsymbol{A} = \boldsymbol{C} + \boldsymbol{D},$$

where $\quad \boldsymbol{C} = (\boldsymbol{c}_{ij})$ with $\boldsymbol{c}_{ii} = 0$, $i = 1, 2, \dots, n$, and $\boldsymbol{c}_{ij} = \boldsymbol{a}_{ij}$ for $i \neq j$;

$\boldsymbol{D} = \operatorname{diag}\{\boldsymbol{d}_1, \boldsymbol{d}_2, \dots, \boldsymbol{d}_n\}$ with $\boldsymbol{d}_i = \boldsymbol{a}_{ii}$, $i = 1, 2, \dots, n$.

Then

$$\mathcal{S}(\boldsymbol{x}) = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\cdot\operatorname{dual}\boldsymbol{x} - \boldsymbol{b},$$

and the main equation takes the form

$$\boldsymbol{C}x + \boldsymbol{D}\cdot\operatorname{dual}x - \boldsymbol{b} = 0.$$

Also, we may suppose that $\boldsymbol{D}$ is regular, i. e. $0 \notin \boldsymbol{d}_i$ for $i = 1, 2, \dots, n$.

Kaucher complete interval arithmetic $\mathbb{KR}$

Duality relations

$$\text{dual}\,(\boldsymbol{x} + \boldsymbol{y}) \;=\; \text{dual}\,\boldsymbol{x} + \text{dual}\,\boldsymbol{y}$$

$$\text{dual}\,(\boldsymbol{x}\,\boldsymbol{y}) \;=\; \text{dual}\,\boldsymbol{x} \cdot \text{dual}\,\boldsymbol{y}$$

Involutions composition

$$-\text{opp}\,\boldsymbol{x} \;=\; \text{dual}\,\boldsymbol{x}$$

$$\text{inv}\,(\text{dual}\,\boldsymbol{x}) \;=\; \boldsymbol{x}^{-1} = [\,1/\overline{\boldsymbol{x}}, 1/\underline{\boldsymbol{x}}]$$

Let $\boldsymbol{x}^*$ be a formal solution to the interval system $\mathcal{S}_i(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{x}) = 0$:

$$\boldsymbol{C}\boldsymbol{x}^* + \boldsymbol{D}\cdot\text{dual}\,\boldsymbol{x}^* - \boldsymbol{b} = 0.$$

Adding $\text{opp}\,(\boldsymbol{D}\cdot\text{dual}\,\boldsymbol{x}^*)$ to both sides gives

$$\text{opp}\,(\boldsymbol{D}\cdot\text{dual}\,\boldsymbol{x}^*) = \boldsymbol{C}\boldsymbol{x}^* - \boldsymbol{b},$$

then multiply the equation by (-1):

$$\text{dual}\,(\boldsymbol{D}\cdot\text{dual}\,\boldsymbol{x}^*) = \boldsymbol{b} - \boldsymbol{C}\boldsymbol{x}^*.$$

Finally, using duality relations, we get

$$(\text{dual}\,\boldsymbol{D})\cdot\boldsymbol{x}^* = \boldsymbol{b} - \boldsymbol{C}\boldsymbol{x}^*. \qquad (*)$$

If the off-diagonal entries of $\boldsymbol{A}$ does not contain zero, then there exist inv (dual $\boldsymbol{D}$), such that

$$\text{inv}\,(\text{dual}\ \boldsymbol{D}) \cdot \text{dual}\ \boldsymbol{D} = I.$$

Moreover, inv dual $(\cdot) = (\cdot)^{-1}$ implies

$$\text{inv}\,(\text{dual}\ \boldsymbol{D})\ =\ \boldsymbol{D}^{-1}\ =\ \text{diag}\Big\{1/\boldsymbol{d}_1, 1/\boldsymbol{d}_2, \ldots, 1/\boldsymbol{d}_n\Big\}.$$

Multiplying both sides of $(*)$ by it, we get

$$x = \boldsymbol{D}^{-1}(\boldsymbol{b} - \boldsymbol{C}x)$$

— an equaivalent fixed-pont form of the main equation $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{x}) = 0$

Enclosure quality

Proposition 7

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ *be an interval H-matrix,* $\boldsymbol{D}, \boldsymbol{C}$ *be diagonal and off-diagonal parts of* $\boldsymbol{A}$ *respectively and* $\boldsymbol{b} \in \mathbb{IR}^{n}$*.*

If $\boldsymbol{x}^*$ *is a formal solution to the interval linear system*

$$\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, x) \; = \; \boldsymbol{C}x + \boldsymbol{D}\cdot\mathrm{dual}\, x - \boldsymbol{b} \; = \; 0,$$

then the following inequality holds:

$$\mathrm{Dist}\left(\square\Xi(\boldsymbol{A}, \boldsymbol{b}),\, \boldsymbol{x}^*\right) \;\leq\; 2\,(I - |\boldsymbol{D}^{-1}\boldsymbol{C}|)^{-1}\,|\boldsymbol{D}^{-1}\boldsymbol{C}|\cdot\mathrm{rad}\left(\square\Xi(\boldsymbol{A}, \boldsymbol{b})\right).$$

Enclosure quality

Corollary

Let the conditions of Proposition 7 hold.

If, for some absolute matrix norm, the value $\eta := \|\boldsymbol{D}^{-1}\boldsymbol{C}\|$ is such that $\eta < 1$, then the inequality

$$\left\| \mathrm{Dist}\left(\square\Xi,\, \boldsymbol{x}^*\right) \right\| \;\leq\; \frac{2\eta}{1-\eta} \cdot \|\mathrm{rad}\,(\square\Xi)\|$$

is true for any consistent absolute vector norm.

Computing formal solutions

M.A. Sainz, E. Gardeñes, L. Jorba

1) Stationary iteration processes:

- we reduce the interval system
  to the fixed-point form $x = \boldsymbol{G}x + \boldsymbol{h}$,

- then take an initial approximation $\boldsymbol{x}^{(0)}$ and launch iterating

$$\boldsymbol{x}^{(k+1)} \leftarrow \boldsymbol{G}\boldsymbol{x}^{(k)} + \boldsymbol{h}, \qquad k = 1, 2, \ldots$$

2) Reducing the solution of the equations system to an optimization problem

Immersion into linear space

$$\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, x) = 0 \qquad\qquad \boldsymbol{C}x = \boldsymbol{d} \qquad\qquad x = \boldsymbol{C}x + \boldsymbol{d}$$

Unfortunately, $\mathbb{KR}^n$ is not a linear space:

the lack of distributivity violates the axiom

$$(\lambda + \mu)\,\boldsymbol{x} = \lambda\,\boldsymbol{x} + \mu\,\boldsymbol{x}$$

in order to use the concepts of differentiability, convexity and so on,
it is necessary to move to a linear space . . .

Immersion into linear space

Any one-one mapping (bijection)

$$\iota : \mathbb{KR}^n \longrightarrow \mathbb{R}^{2n}$$

induces a one-one mapping of

$$\text{the set of the mappings over } \mathbb{KR}^n$$
$$\downarrow$$
$$\text{the set of the mappings over } \mathbb{R}^{2n}$$

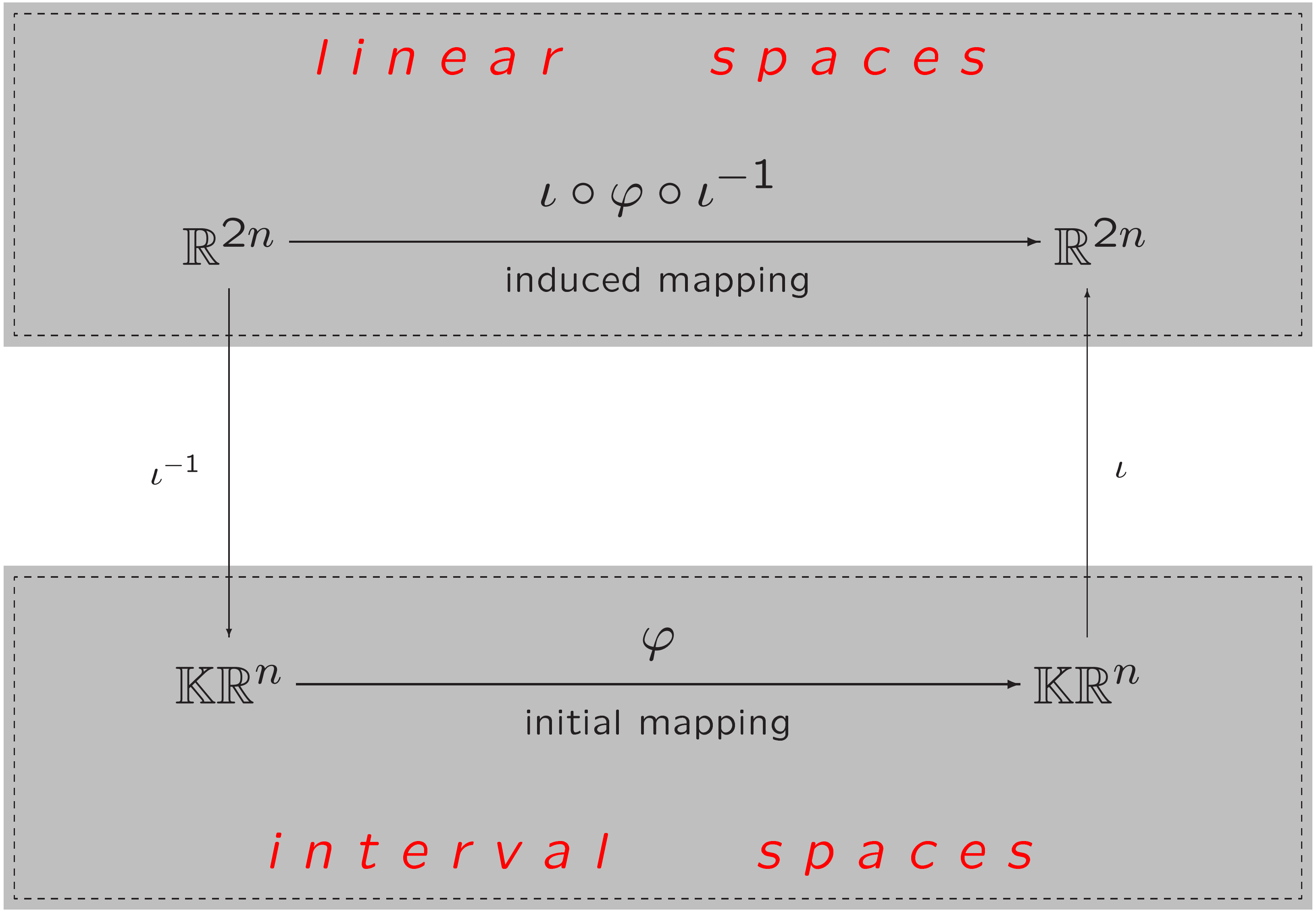
A unique *induced mapping*

$$\iota \circ \phi \circ \iota^{-1} : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$$

corresponds to every mapping

$$\phi : \mathbb{KR}^n \longrightarrow \mathbb{KR}^n$$

Immersion into linear space



Immersion to linear space

If $\iota(0) = 0$, then we can change
solving an equation in $\mathbb{KR}^n$ to solving an equation in $\mathbb{R}^{2n}$.

A compromise:

simplicity of the immersion $\leftrightarrow$ convenient form of the induced maps ?

Definition

The immersion sti $: \mathbb{KR}^n \to \mathbb{R}^{2n}$ *that acts according to the rule*

$$(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n) \;\mapsto\; (-\underline{\boldsymbol{x}}_1, -\underline{\boldsymbol{x}}_2, \dots, -\underline{\boldsymbol{x}}_n, \overline{\boldsymbol{x}}_1, \overline{\boldsymbol{x}}_2, \dots, \overline{\boldsymbol{x}}_n),$$

will be called *standard immersion* *of* $\mathbb{KR}^n$ *into* $\mathbb{R}^{2n}$.

Induced equation

Instead of

$$\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, x) = 0 \qquad \text{in } \mathbb{KR}^n,$$

we are going to solve the induced equation in $\mathbb{R}^{2n}$:

$$\mathfrak{S}(y) = 0$$

$$\mathfrak{S}_i(y) = -\underline{\left(\boldsymbol{a}_{ii} \cdot \operatorname{dual} \boldsymbol{x}_i + \sum_{j \neq i} \boldsymbol{a}_{ij} \boldsymbol{x}_j - \boldsymbol{b}_i\right)}, \qquad i = 1, 2, \dots, n,$$

$$\mathfrak{S}_i(y) = \overline{\left(\boldsymbol{a}_{ii} \cdot \operatorname{dual} \boldsymbol{x}_i + \sum_{j \neq i} \boldsymbol{a}_{ij} \boldsymbol{x}_j - \boldsymbol{b}_i\right)}, \qquad i = n+1, \dots, 2n.$$

Investigating the induced equation

Proposition 4

The induced mapping $\mathfrak{S}: \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ is continuous.

Investigating the induced equation

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The induced mapping $\mathfrak{S} : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ is continuous.

What about differentiability, smoothness, etc.?

Investigating the induced equation

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The induced mapping $\mathfrak{S} : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ is continuous.

What about differentiability, smoothness, etc.?

We do not have them.

Investigating the induced equation

Proposition 4
The induced mapping $\mathfrak{G} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is continuous.

What about differentiability, smoothness, etc.?

We do not have them. But, instead, we have CONVEXITY !!!

due to subdistributivity

$$\boldsymbol{x}(\boldsymbol{y} + \boldsymbol{z}) \subseteq \boldsymbol{x}\boldsymbol{y} + \boldsymbol{x}\boldsymbol{z}$$

for proper $\boldsymbol{x}$

Plan of further actions

Subdistributivity implies convexity of the function $\mathfrak{S}(x)$ with respect to the components-wise order "$\leq$"

$$\Downarrow$$

Existence of the nonempty subdifferential $\partial_{\leq}\mathfrak{S}(x)$, easily computable, since $\mathfrak{S}(x)$ is polyhedral function

$$\Downarrow$$

Having the subdifferential $\partial_{\leq}\mathfrak{S}(x)$ allows us to construct subdifferential Newton method

Order convexity

Definition

Let a partial order "$\preccurlyeq$" be defined on $\mathbb{R}^q$. The map $F : \mathbb{R}^p \to \mathbb{R}^q$ is called order convex with respect to "$\preccurlyeq$", if

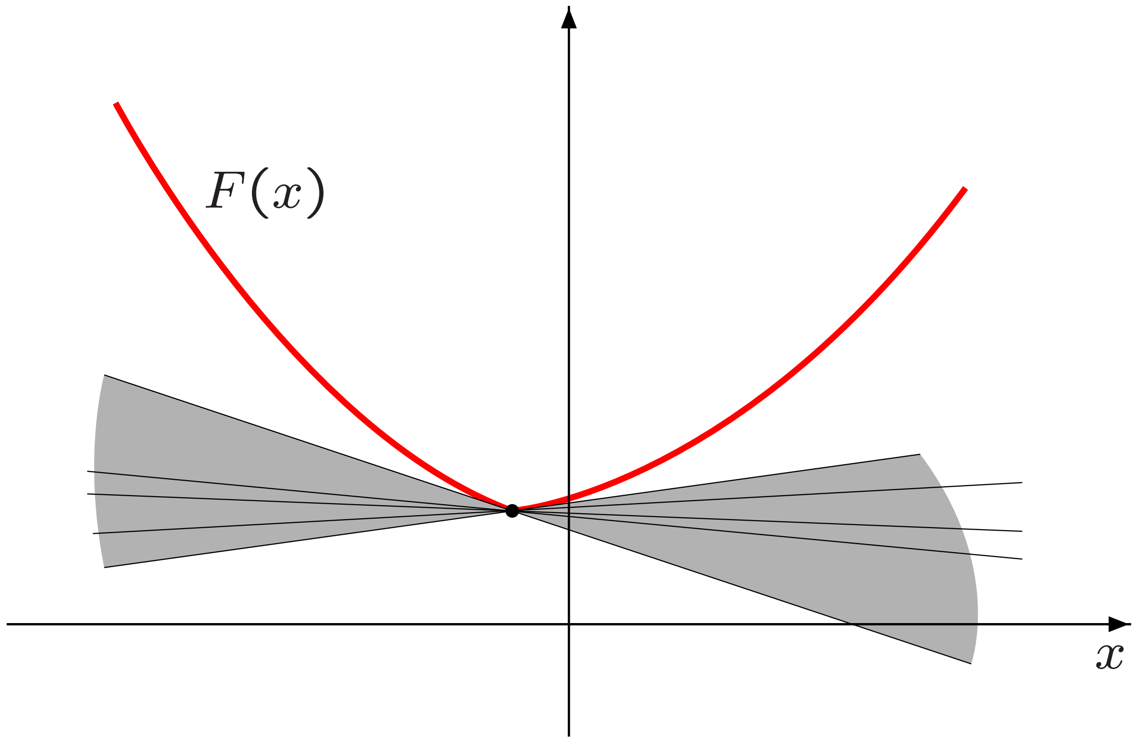
$$F(\lambda y + (1-\lambda)z) \preccurlyeq \lambda F(y) + (1-\lambda)F(z)$$

for every y, $z \in \mathbb{R}^p$ and $\lambda \in (0, 1)$.

Theorem

The induced mapping $\mathfrak{S}(y)$ is order convex on $\mathbb{R}^{2n}$ with respect to the component-wise partial order "$\leq$".

Subgradients and subdifferential of convex functions

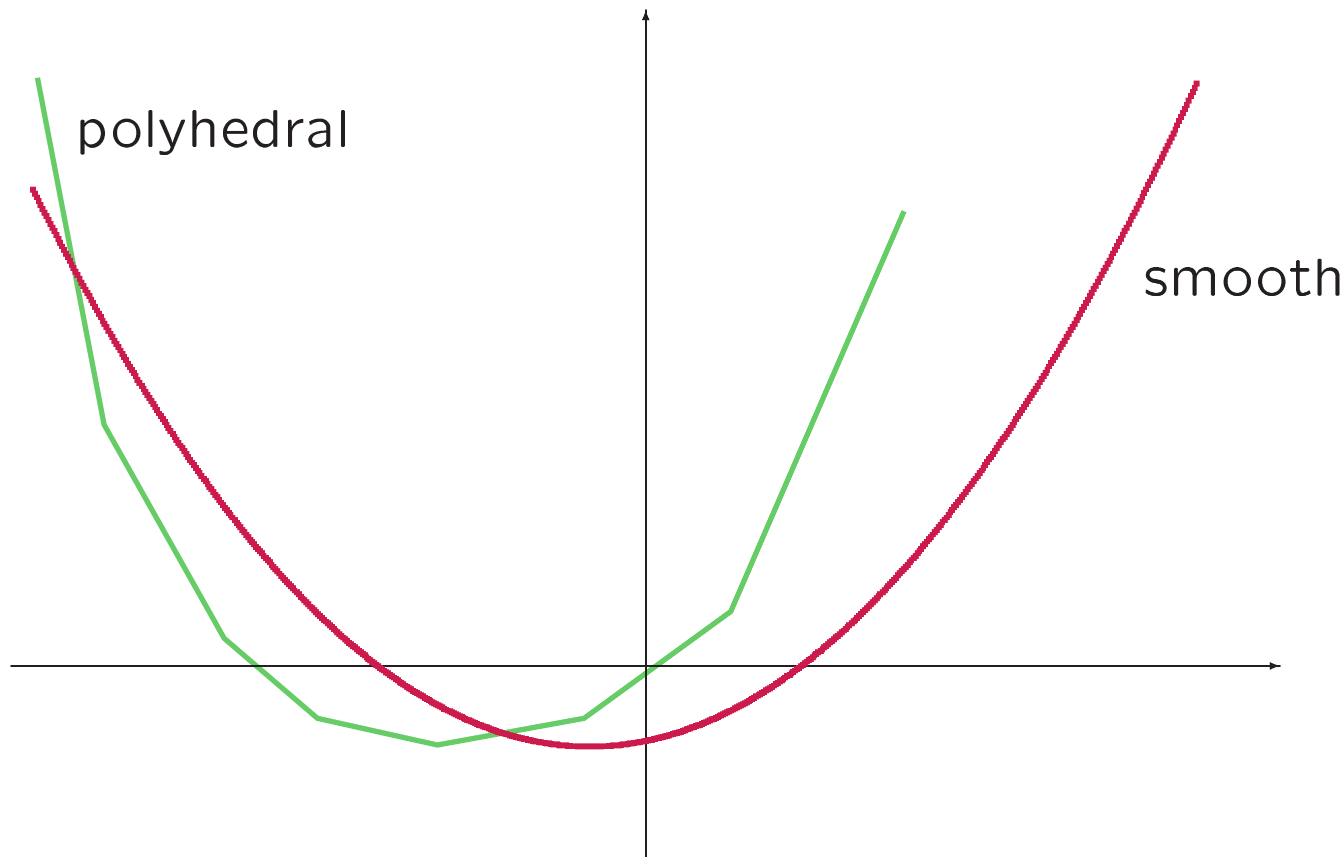


Subdifferential of the map F in $x \in \mathbb{R}^p$ is a set $\partial_{\preccurlyeq} F(x)$ formed by all such linear operators $D : \mathbb{R}^p \longrightarrow \mathbb{R}^q$ that

$$D(v) \preccurlyeq F(x+v) - F(v)$$

for every $v \in \mathbb{R}^p$. Its elements are subgradients of F at the point x.

Polyhedrality



Theorem

The coordinate components $\mathfrak{S}_i(x)$, $i = 1, 2, \ldots, 2n$, are convex polyhedral functions $\mathbb{R}^{2n} \to \mathbb{R}$.

Subdifferential Newton method

for the solution of the equations system $\mathfrak{S}(x) = 0$ in $\mathbb{R}^{2n}$

Take an initial approximation $x^{(0)} \in \mathbb{R}^{2n}$.

If the $(k-1)$-th iteration $x^{(k-1)} \in \mathbb{R}^{2n}$, $k = 1, 2, \ldots$, is found, compute a subgradient $D^{(k-1)}$ of the map $\mathfrak{S}$ at the point $x^{(k-1)}$ and set

$$x^{(k)} \leftarrow x^{(k-1)} - \tau \left(D^{(k-1)} \right)^{-1} \mathfrak{S}\left(x^{(k-1)} \right),$$

where $\tau \in \,]0, 1]$ is a damping factor.

Subdifferential Newton method

Convergence theorem

Let the interval $n \times n$-matrix $\boldsymbol{A}$ be "sufficiently narrow" and the interval $2n \times 2n$-matrix

$$\begin{pmatrix} \boldsymbol{A}^{+} & \boldsymbol{A}^{-} \\ \boldsymbol{A}^{-} & \boldsymbol{A}^{+} \end{pmatrix}$$

is regular. Then, for $\tau = 1$, subdifferential Newton method converges in a finite number of steps to $\mathrm{sti}\,(\boldsymbol{x}^{*})$, *where $\boldsymbol{x}^{*}$ is a formal solution to the equation* $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, x) = 0$.

Subdifferential Newton method

In practice, we recommend to take $\tau = 1$ at first.

Then, in case of convergence, subdifferential Newton method produces *exact* solution in a few iterations (usually, $\leq n$).

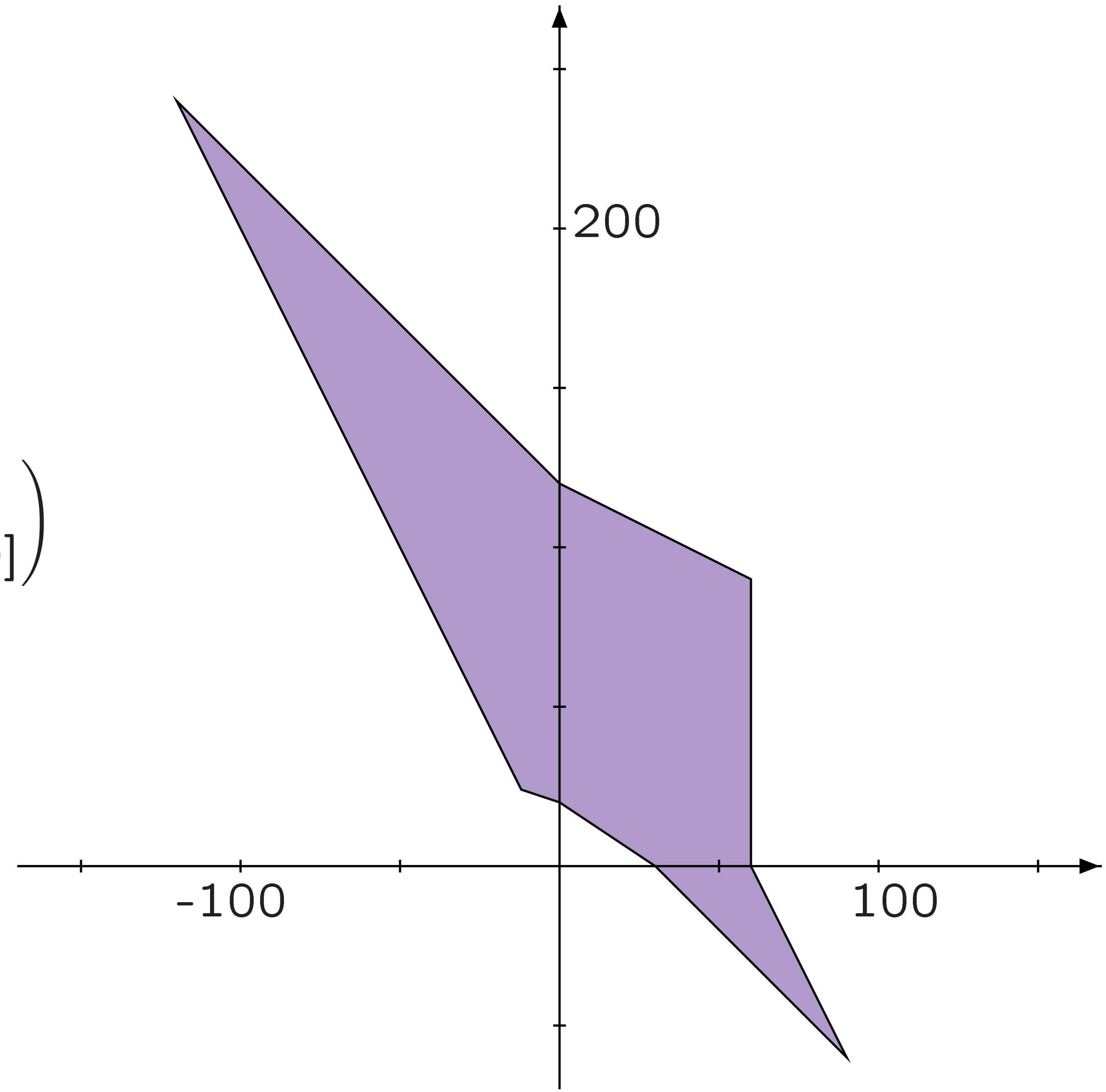
The reason is polyhedrality of the functions involved

If the method does not converge, we recommend to decrease τ.

Practical complexity $\approx O(n^3)$
with a small constant . . .

Example — Hansen system

$$\begin{pmatrix} [2,3] & [0,1] \\ [1,2] & [2,3] \end{pmatrix} x = \begin{pmatrix} [0,120] \\ [60,240] \end{pmatrix}$$



Example

For the interval Hansen system

$$\begin{pmatrix} [2,3] & [0,1] \\ [1,2] & [2,3] \end{pmatrix} x = \begin{pmatrix} [0,120] \\ [60,240] \end{pmatrix},$$

formal solution of the coresponding equation

$$\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, x) = 0$$

is the box

$$\begin{pmatrix} [-120,90] \\ [-60,240] \end{pmatrix},$$

— optimal enclosure of the solution set

Subdifferential Newton method

computes it in 3 (three) iterations

Resume

Based on Miranda theorem, we can derive one more version of formal (algebraic) approach to enclosing solution sets for interval linear systems.

If subdifferential Newton method is used for the computation of formal solutions, we get a technique whose advantages are

- no additional transformations are necessary
- high computational efficacy.

Its disadvantage (that may be easily fixed) is

- the absence of “verification” in the original form.

I appreciate your attention